

# NTRU+: Compact Construction of NTRU Using Simple Encoding Method

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| <b>Authors</b>        | Jong Hwan Park (Sangmyung University)<br>Jonghyun Kim (Korea University)   |
| <b>Contact</b>        | <a href="mailto:jhpark@smu.ac.kr">jhpark@smu.ac.kr</a>   |
| <b>Homepage</b>       | <a href="https://www.ntruplus.org">https://www.ntruplus.org</a>  |
| <b>Reference Code</b> | <a href="https://www.ntruplus.org/specification">https://www.ntruplus.org/specification</a><br>Integrity Hash (SHA-256):<br>44fa5b5b 6edbb284 9322f41e 6b0e8088<br>74caa9f6 efcaa83b a72dce0d 47512615 |

# NTRU+: Compact Construction of NTRU Using Simple Encoding Method

Jonghyun Kim\* Jong Hwan Park†

January 30, 2026

## Abstract

NTRU was the first practical public-key encryption scheme based on lattice problems over polynomial rings and has remained resilient against cryptanalysis for several decades. However, classical NTRU and its variants face several limitations, such as difficulty in achieving a negligible worst-case correctness error with a moderate modulus, complex message sampling such as fixed hamming weight sampling, and relatively slower performance compared to other lattice-based schemes.

In this work, we propose a new NTRU-based key encapsulation mechanism, called NTRU+, which addresses the aforementioned drawbacks. NTRU+ is constructed by sequentially applying two generic transformations,  $\text{ACWC}_2$  and  $\overline{\text{FO}}^\perp$  (a variant of the Fujisaki–Okamoto transformation), where the former enables negligible worst-case correctness error and the latter enables chosen-ciphertext security without requiring re-encryption. Both  $\text{ACWC}_2$  and  $\overline{\text{FO}}^\perp$  leverage a randomness-recovery algorithm unique to NTRU and a novel message-encoding method called the semi-generalized one-time pad (SOTP). In particular, SOTP supports messages sampled from natural bit-string spaces with arbitrary distributions. We provide three parameter sets for NTRU+ and present implementation results using NTT-friendly rings over cyclotomic trinomials.

**Keywords:** NTRU, RLWE, Lattice-based cryptography, Post-quantum cryptography.

## 1 Introduction

The NTRU encryption scheme [19] was introduced in 1998 by Hoffstein, Pipher, and Silverman as the first practical public-key encryption scheme based on lattice problems over polynomial rings. Its security relies on the NTRU problem [19], which has remained resilient against significant cryptanalytic attacks for over two decades. This longer history, compared to other lattice-based problems such as Ring-LWE and Module-LWE, has been regarded as an important factor in NTRU being selected as a finalist in the NIST PQC standardization process. Although the finalist NTRU [10] was not selected by NIST among the first four quantum-resistant cryptographic algorithms, it still offers several distinct advantages over other lattice-based schemes such as KYBER [34] and Saber [13]. Specifically, NTRU provides (1) a compact ciphertext structure consisting of a single polynomial and (2) potentially faster encryption and decryption without requiring coefficient sampling for the public key polynomial.

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\*Korea University, Seoul, Korea. Email: yoswuk@korea.ac.kr.

†Sangmyung University, Seoul, Korea. Email: jhpark@smu.ac.kr.

The central design principle of NTRU is described over the ring  $R_q = \mathbb{Z}_q[x]/\langle f(x) \rangle$ , where  $q$  is a positive integer and  $f(x)$  is a polynomial. The public key is generated as  $\mathbf{h} = pg/(pf' + 1) \in R_q$ <sup>1</sup>, where  $g$  and  $f'$  are sampled from a narrow distribution  $\psi$ , and  $p$  is a positive integer that is smaller than and co-prime to  $q$  (e.g.,  $p = 3$ ). The corresponding private key is  $\mathbf{f} = pf' + 1$ . To encrypt a message  $m$  sampled from the message space  $\mathcal{M}'$ , one samples two polynomials  $\mathbf{r}$  and  $\mathbf{m}$ , with coefficients drawn from the distribution  $\psi$ , and computes the ciphertext  $\mathbf{c} = \mathbf{h}\mathbf{r} + \mathbf{m}$  in  $R_q$ . An (efficient) encoding method may be used to encode  $m \in \mathcal{M}'$  into  $\mathbf{m}$  and  $\mathbf{r} \in R_q$ . Alternatively, one may directly sample  $\mathbf{m}$  and  $\mathbf{r}$  from  $\psi$ , where  $\mathbf{m}$  is regarded as the message to be encrypted. To decrypt the ciphertext  $\mathbf{c}$ , one computes  $\mathbf{c}\mathbf{f}$  in  $R_q$ , recovers  $\mathbf{m}$  by deriving the value  $\mathbf{c}\mathbf{f}$  modulo  $p$ , and (if necessary) decodes  $\mathbf{m}$  to obtain the message  $m$ . The decryption of NTRU works correctly if all the coefficients of the polynomial  $p(\mathbf{g}\mathbf{r} + \mathbf{f}'\mathbf{m}) + \mathbf{m}$  are less than  $q/2$ . Otherwise, the decryption fails, and the probability of failure is referred to as the *correctness* (or *decryption*) *error*.

In the context of chosen-ciphertext attacks, NTRU, like other public-key encryption schemes, must guarantee a negligible worst-case correctness error. This requirement is essential to prevent the leakage of secret key information through adversarial decryption queries, as observed in attacks on lattice-based encryption schemes [12, 22]. Roughly speaking, the worst-case correctness error refers to the maximum probability of decryption failure taken over all possible messages. It reflects the possibility that an adversary  $\mathcal{A}$  may *maliciously* craft messages and randomness rather than sampling them from their intended distributions. In the case of NTRU, a decryption failure on a ciphertext  $\mathbf{c} = \mathbf{h}\mathbf{r} + \mathbf{m}$  informs  $\mathcal{A}$  that at least one coefficient of  $p(\mathbf{g}\mathbf{r} + \mathbf{f}'\mathbf{m}) + \mathbf{m}$  is greater than or equal to  $q/2$ . Therefore, if  $\mathcal{A}$  has control over  $\mathbf{r}$  and  $\mathbf{m}$ , even a single decryption failure may leak information about  $\mathbf{g}$  and  $\mathbf{f}'$ .

When designing NTRU, two approaches can be used to achieve negligible worst-case correctness error. One is to draw  $\mathbf{m}$  and  $\mathbf{r}$  directly from the distribution  $\psi$  while setting the modulus  $q$  to be relatively large. Choosing a larger  $q$  guarantees, with high probability, that all coefficients of  $p(\mathbf{g}\mathbf{r} + \mathbf{f}'\mathbf{m}) + \mathbf{m}$  are less than  $q/2$  for *nearly all* inputs  $\mathbf{m}$  and  $\mathbf{r}$ , although this comes at the cost of increased key and ciphertext sizes. This approach is used by the third-round finalist NTRU [10], whose parameters achieve *perfect* correctness (i.e., the worst-case correctness error becomes zero for all possible  $\mathbf{m}$  and  $\mathbf{r}$ ). In contrast, the other approach [16] uses an encoding method in which a message  $m \in \mathcal{M}'$  is used as a seed to sample  $\mathbf{m}$  and  $\mathbf{r}$  according to  $\psi$ . Under the Fujisaki–Okamoto (FO) transformation [17], decrypting a ciphertext  $\mathbf{c}$  requires re-encrypting  $m$  by following the same sampling process used in encryption. Thus, any ill-formed ciphertext that violates the sampling rule will always fail to decrypt, implying that  $\mathbf{m}$  and  $\mathbf{r}$  must be *honestly* sampled by  $\mathcal{A}$  according to  $\psi$ . Consequently, by preventing  $\mathcal{A}$  from controlling  $\mathbf{m}$  and  $\mathbf{r}$ , the NTRU scheme with the encoding method achieves a worst-case correctness error that is close to its average-case correctness error.

Based on the above observation, [16] proposed generic (average-case to worst-case) transformations<sup>2</sup> that make the average-case correctness error of an underlying scheme close to the worst-case error of the transformed scheme. One of their transformations (denoted by ACWC) is based on an encoding method called the generalized one-time pad (denoted by GOTP). Roughly speaking, ACWC works as follows: a message  $m \in \mathcal{M}'$  is first used to sample  $\mathbf{r}$  and  $\mathbf{m}_1$  according to  $\psi$ , and then  $\mathbf{m}_2 = \text{GOTP}(m, \mathbf{G}(\mathbf{m}_1))$  is computed using a hash function  $\mathbf{G}$ . Finally,  $\mathbf{m}$  is constructed as  $\mathbf{m}_1 \parallel \mathbf{m}_2$ . If GOTP acts as a sampling function such that its output follows  $\psi$ , then  $\mathbf{m}$  and  $\mathbf{r}$  are generated from  $m$  according to  $\psi$ , which can be verified during decryption using the FO transformation. Specifically, for two inputs  $m$  and  $\mathbf{G}(\mathbf{m}_1)$  that are sampled from  $\{-1, 0, 1\}^\lambda$  for some integer  $\lambda$ ,  $\mathbf{m}_2 \in \{-1, 0, 1\}^\lambda$  is computed by doing the component-wise addition modulo 3 on the ternary strings  $m$  and  $\mathbf{G}(\mathbf{m}_1)$ . Thus, if  $\mathbf{G}(\mathbf{m}_1)$  follows a uniformly random distribution over  $\{-1, 0, 1\}^\lambda$ , then  $m$  is hidden in  $\mathbf{m}_2$  due to the one-time pad property.

<sup>1</sup> Alternatively, the public key can be generated as  $\mathbf{h} = pg/f$ , but we use  $\mathbf{h} = pg/(pf' + 1)$  for more efficient decryption.

<sup>2</sup>They proposed two transformations, ACWC<sub>0</sub> and ACWC, but we focus on ACWC, which does not increase the ciphertext size.

| Scheme                               | NTRU[10]   | NTRU-B [16]                         | NTRU+                               |
|--------------------------------------|--|-------------------------------------|-------------------------------------|
| NTT-friendly                         | No   | Yes                                 | Yes                                 |
| Correctness error                    | Perfect  | Worst-case                          | Worst-case                          |
| $(\mathbf{m}, \mathbf{r})$ -encoding | No   | Yes                                 | Yes                                 |
| Message set                          | $\mathbf{m}, \mathbf{r} \leftarrow \{-1, 0, 1\}^n$ | $m \leftarrow \{-1, 0, 1\}^\lambda$ | $m \leftarrow \{0, 1\}^n$           |
| Message distribution                 | Uniform/Fixed-weight                               | Uniform                             | Arbitrary                           |
| CCA transform                        | DPKE + SXV variant                                 | ACWC + FO <sup>⊥</sup>              | ACWC <sub>2</sub> + FO <sup>⊥</sup> |
| Assumptions                          | NTRU, RLWE   | NTRU, RLWE                          | NTRU, RLWE                          |
| Tight reduction                      | Yes  | No                                  | Yes                                 |

$n$ : polynomial degree of the ring.  $\lambda$ : length of the message. DPKE: deterministic public-key encryption.

SXV variant: SXV transformation [33] described in the NTRU finalist.

Table 1: Comparison to previous NTRU constructions

However, an ACWC based on GOTP has two disadvantages in terms of security reduction and message distribution. First, [16] showed that ACWC converts a one-way CPA (OW-CPA) secure underlying scheme into a transformed one that remains OW-CPA secure, although the security reduction is loose<sup>3</sup> and causes an additional loss factor of  $q_G$ , the number of random oracle queries. Second, ACWC requires that even the message  $m \in \mathcal{M}'$  follow a specific distribution because the security analysis of ACWC requires GOTP to have the additional randomness-hiding property, meaning that  $G(\mathbf{m}_1)$  should also be hidden from the output  $\mathbf{m}_2$ . Indeed, the NTRU instantiation from ACWC, called ‘NTRU-B’ [16], requires that  $m$  be chosen uniformly at random from  $\mathcal{M}' = \{-1, 0, 1\}^\lambda$ . Notably, it is difficult to generate exactly uniformly random samples from  $\{-1, 0, 1\}$  in constant time due to rejection sampling. Therefore, it was an open problem [16] to construct a new transformation that permits a different, more easily sampled distribution of a message while relying on the same security assumptions.

## 1.1 Our Results

We propose a new practical NTRU construction called ‘NTRU+’ that addresses the two drawbacks of the previous ACWC. To achieve this, we introduce a new generic ACWC transformation, denoted as ACWC<sub>2</sub>, which utilizes a simple encoding method. By using ACWC<sub>2</sub>, NTRU+ achieves a worst-case correctness error close to the average-case error of the underlying NTRU. Additionally, NTRU+ requires the message  $m$  to be drawn from  $\mathcal{M}' = \{0, 1\}^n$  (for a polynomial degree  $n$ ), following an *arbitrary* distribution with high min-entropy, and is proven to be *tightly* secure under the same assumptions as NTRU-B, the NTRU and RLWE assumptions. To achieve chosen-ciphertext security, NTRU+ relies on a novel FO-equivalent transform without re-encryption, which makes the decryption algorithm of NTRU+ faster than in the ordinary FO transform. In terms of efficiency, we use the idea from [31] to apply the Number Theoretic Transform (NTT) to NTRU+ and therefore instantiate NTRU+ over a ring  $R_q = \mathbb{Z}_q[x]/\langle f(x) \rangle$ , where  $f(x) = x^n - x^{n/2} + 1$  is a cyclotomic trinomial. By selecting appropriate  $(n, q)$  and  $\psi$ , we suggest three parameter sets for NTRU+ and provide the implementation results for NTRU+ in each parameter set. Table 1 lists the main differences between the previous NTRU constructions [10, 16] and NTRU+. In the following section, we describe our technique, focusing on these differences.

<sup>3</sup>[16] introduced the  $q$ -OW-CPA security notion, where an adversary outputs a set  $Q$  of size at most  $q$  and wins if the correct message corresponding to a challenged ciphertext is in  $Q$ . We believe that  $q$ -OW-CPA leads to a security loss of  $q$ .

|                      | ACWC <sub>0</sub> [16]       | ACWC[16]                    | ACWC <sub>2</sub>                 |
|----------------------|------------------------------|-----------------------------|-----------------------------------|
| Message encoding     | No                           | GOTP                        | SOTP                              |
| Message distribution | Arbitrary                    | Uniform                     | Arbitrary                         |
| Ciphertext expansion | Yes                          | No                          | No                                |
| Transformation       | OW-CPA $\rightarrow$ IND-CPA | OW-CPA $\rightarrow$ OW-CPA | OW-CPA $\rightarrow$ IND-CPA      |
| Tight reduction      | No                           | No                          | Yes                               |
| Underlying PKE       | Any                          | Any                         | Injective + RR +<br>AC-MR + VC-MR |

{AC, VC}-MR: {arbitrary ciphertext, valid ciphertext} message-recoverable. RR: randomness-recoverable.

Table 2: Comparison to previous ACWC transformations

**ACWC<sub>2</sub> Transformation with Tight Reduction.** ACWC<sub>2</sub> is a new generic transformation that enables the aforementioned conversion from average-case to worst-case correctness error. To apply ACWC<sub>2</sub>, the underlying scheme must satisfy injectivity, arbitrary-ciphertext and valid-ciphertext message-recoverability (AC-MR and VC-MR), and randomness-recoverability (RR) properties that are all inherent to NTRU.<sup>4</sup> Additionally, ACWC<sub>2</sub> introduces an encoding method called the semi-generalized one-time pad (denoted by SOTP). In contrast to the ACWC in [16], ACWC<sub>2</sub> equipped with SOTP = (Encode, Decode) works as follows: first, a message  $m \in \mathcal{M}'$  is used to sample  $\mathbf{r}$  according to  $\psi$ , and then  $\mathbf{m} = \text{Encode}(m, G(\mathbf{r}))$  is computed, where the coefficients follow  $\psi$ , using a hash function  $G$ . When decrypting a ciphertext  $\mathbf{c} = \text{Enc}(pk, \mathbf{m}; \mathbf{r})$  under a public key  $pk$ ,  $\mathbf{m}$  is recovered by the usual decryption algorithm, and using  $\mathbf{m}, \mathbf{r}$  is also recovered by a randomness-recovery algorithm. Finally, applying Decode to  $\mathbf{m}$  and  $G(\mathbf{r})$  yields the original message  $m$ .

The VC-MR property of the underlying scheme allows ACWC<sub>2</sub> to transform an OW-CPA secure scheme into an IND-CPA secure one without any significant security loss. The proof idea is straightforward: unless an IND-CPA adversary  $\mathcal{A}$  queries the target randomness  $\mathbf{r}$  to the (classical) random oracle  $G$ ,  $\mathcal{A}$  obtains no information about the challenge message  $m_b$  due to the message-hiding property of SOTP. However, for each query  $\mathbf{r}_i$  made by  $\mathcal{A}$  to  $G$  ( $i = 1, \dots, q_G$ ), a reductionist can verify whether  $\mathbf{r}_i$  corresponds to the OW-CPA challenge ciphertext by using the message-recovery algorithm. Consequently, the reductionist can identify the exact  $\mathbf{r}_i$  among the  $q_G$  queries if  $\mathcal{A}$  has queried it to  $G$ . In this security analysis, it is sufficient for SOTP to satisfy only the message-hiding property, making it simpler than GOTP, which must ensure both message-hiding and randomness-hiding.

Table 2 compares the previous ACWC transformations with our new ACWC<sub>2</sub>. In addition to ACWC based on GOTP, [16] proposed another generic transformation, denoted by ACWC<sub>0</sub>, which does not use any message-encoding method. In ACWC<sub>0</sub>, a bit-string message  $m$  is encrypted as  $(\text{Enc}(pk, \mathbf{m}; \mathbf{r}), F(\mathbf{m}) \oplus m)$  using a hash function  $F$ . This approach causes ciphertext expansion due to the additional term  $F(\mathbf{m}) \oplus m$ , a redundancy that does not appear in either ACWC or ACWC<sub>2</sub>. Like ACWC<sub>2</sub>, ACWC<sub>0</sub> transforms an OW-CPA secure scheme into an IND-CPA secure one, but its security reduction is not as tight as that of ACWC<sub>2</sub>. Furthermore, neither ACWC<sub>0</sub> nor ACWC<sub>2</sub> requires any specific message distribution, whereas ACWC requires  $m \in \mathcal{M}'$  to be sampled uniformly from  $\mathcal{M}'$ . In terms of applicability, while ACWC<sub>0</sub> and ACWC are applicable to any OW-CPA secure scheme, ACWC<sub>2</sub> applies specifically to those that additionally satisfy injectivity, AC-MR, VC-MR, and RR.

<sup>4</sup>In the decryption of NTRU with  $pk = \mathbf{h}$ , given  $(pk, \mathbf{c}, \mathbf{m})$ , the randomness  $\mathbf{r}$  is recovered as  $\mathbf{r} = (\mathbf{c} - \mathbf{m})\mathbf{h}^{-1}$ . Similarly, given  $(pk, \mathbf{c}, \mathbf{r})$ , the message  $\mathbf{m}$  is recovered as  $\mathbf{m} = \mathbf{c} - \mathbf{r}\mathbf{h}$ .

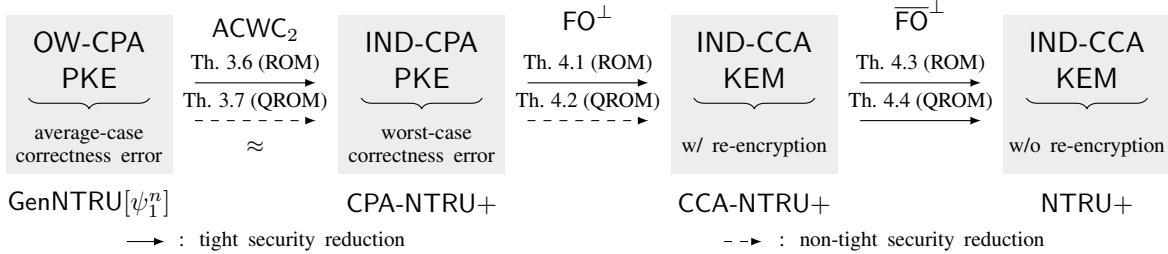


Figure 1: Overview of security reductions for KEM

**FO-Equivalent Transform without Re-encryption.** To achieve chosen-ciphertext (IND-CCA) security, we apply the generic transform  $\text{FO}^\perp$  to the  $\text{ACWC}_2$ -derived scheme, which is IND-CPA secure. As with other FO-transformed schemes, the resulting scheme from  $\text{ACWC}_2$  and  $\text{FO}^\perp$  is still required to perform re-encryption in the decryption process to check if (1)  $(\mathbf{m}, \mathbf{r})$  are correctly generated from  $m$  and (2) a (decrypted) ciphertext  $\mathbf{c}$  is correctly encrypted from  $(\mathbf{m}, \mathbf{r})$ . However, by using the RR property of the underlying scheme, we further remove the re-encryption process from  $\text{FO}^\perp$ . Instead, the more advanced transform (denoted by  $\overline{\text{FO}}^\perp$ ) simply checks whether  $\mathbf{r}$  from the randomness-recovery algorithm is the same as the (new) randomness  $\mathbf{r}'$  created from  $m$ . We show that  $\overline{\text{FO}}^\perp$  is functionally identical to  $\text{FO}^\perp$  by proving that the randomness-checking process in  $\overline{\text{FO}}^\perp$  is equivalent to the re-encryption process  $\text{FO}^\perp$ . The equivalence proof relies mainly on the randomness-recoverability of the underlying schemes. As a result, although the RR property seems to incur some additional decryption cost, it ends up making the decryption algorithm faster than the original FO transform. Figure 1 presents an overview of security reductions from OW-CPA to IND-CCA.

**Simple SOTP Instantiation with More Convenient Sampling Distributions.** As mentioned previously,  $\text{ACWC}_2$  is based on an efficient construction of  $\text{SOTP} = (\text{Encode}, \text{Decode})$  that takes  $m$  and  $G(\mathbf{r})$  as inputs and outputs  $\mathbf{m} = \text{Encode}(m, G(\mathbf{r}))$ . In particular, computing  $\mathbf{m} = \text{Encode}(m, G(\mathbf{r}))$  requires that each coefficient of  $\mathbf{m}$  should follow  $\psi$ , while preserving the message-hiding property. To achieve this, we set  $\psi$  as the centered binomial distribution (CBD)  $\psi_k$  with  $k = 1$ , which is easily obtained by subtracting two uniformly random bits from each other. For a polynomial degree  $n$  and hash function  $G : \{0, 1\}^* \rightarrow \{0, 1\}^{2n}$ ,  $m$  is chosen from the message space  $\mathcal{M}' = \{0, 1\}^n$  for an arbitrary distribution (with high min-entropy) and  $G(\mathbf{r}) = y_1 \parallel y_2 \in \{0, 1\}^n \times \{0, 1\}^n$ . SOTP then computes  $\tilde{m} = (m \oplus y_1) - y_2$  by bit-wise subtraction and assigns each subtraction value of  $\tilde{m}$  to the coefficient of  $\mathbf{m}$ . By the one-time pad property, it is easily shown that  $m \oplus y_1$  becomes uniformly random in  $\{0, 1\}^n$  (and thus message-hiding) and each coefficient of  $\mathbf{m}$  follows  $\psi_1$ . Since  $\mathbf{r}$  is also sampled from a hash value of  $m$  according to  $\psi_1$ , all sampling distributions in NTRU+ are easy to implement. We can also expect that, similar to the case of  $\psi_1$ , the SOTP is expanded to sample a centered binomial distribution reduced modulo 3 (i.e.,  $\overline{\psi}_2$ ) by summing up and subtracting more uniformly random bits.

**NTT-Friendly Rings Over Cyclotomic Trinomials.** NTRU+ is instantiated over a polynomial ring  $R_q = \mathbb{Z}_q[x]/\langle f(x) \rangle$ , where  $f(x) = x^n - x^{n/2} + 1$  is a cyclotomic trinomial of degree  $n = 2^i 3^j$ . [31] showed that, with appropriate parameterization of  $n$  and  $q$ , such a ring can also provide NTT operation essentially as fast as that over a ring  $R_q = \mathbb{Z}_q[x]/\langle x^n + 1 \rangle$ . Moreover, because the choice of a cyclotomic trinomial is moderate, it provides more flexibility to satisfy a certain level of security. Based on these results, we choose three parameter sets for NTRU+, where the polynomial degree  $n$  of  $f(x) = x^n - x^{n/2} + 1$  is set to be 768,

864, and 1152, and the modulus  $q$  is 3457 for all cases. Table 7 lists the comparison results between finalist NTRU [10], KYBER [34], and NTRU+ in terms of security and efficiency. To estimate the concrete security level of NTRU+, we use the Lattice estimator [1] for the RLWE problem and the NTRU estimator [10] for the NTRU problem, considering that all coefficients of each polynomial  $f'$ ,  $g$ ,  $r$ , and  $m$  are drawn according to the centered binomial distribution  $\psi_1$ . The implementation results in Table 7 are estimated with reference and AVX2 optimizations. We can observe that NTRU+ outperforms NTRU at a similar security level.

## 1.2 Related Works

The first-round NTRUEncrypt [36] submission to the NIST PQC standardization process was an NTRU-based encryption scheme with the NAEP padding method [23]. Roughly speaking, NAEP is similar to our SOTP, but the difference is that it does not completely encode  $m$  to prevent an adversary  $\mathcal{A}$  from choosing  $m$  maliciously. This is due to the fact that  $m := \text{NAEP}(m, G(\mathbf{hr}))$  is generated by subtracting two  $n$ -bit strings  $m$  and  $G(\mathbf{hr})$  from each other, i.e.,  $m - G(\mathbf{hr})$  by bit-wise subtraction, and then assigning them to the coefficients of  $m$ . Since  $m$  can be maliciously chosen by  $\mathcal{A}$  in NTRUEncrypt,  $m$  can also be maliciously chosen, regardless of  $G(\mathbf{hr})$ .

The finalist NTRU [10] was submitted as a key encapsulation mechanism (KEM) that provides four parameter sets for perfect correctness. To achieve chosen-ciphertext security, [10] relied on a variant of the SXV [33] conversion, which also avoids re-encryption during decapsulation. Similar to NTRU+, the SXV variant requires the rigidity [7] of an underlying scheme and uses the notion of deterministic public-key encryption (DPKE) where  $(m, r)$  are all recovered as a message during decryption. In the NTRU construction, the recovery of  $r$  is conceptually the same as the existence of the randomness-recovery algorithm RRec. Instead of removing re-encryption, the finalist NTRU needs to check whether  $(m, r)$  are selected correctly from predefined distributions.

In 2019, Lyubashevsky et al. [31] proposed an efficient NTRU-based KEM called NTTRU by applying NTT to the ring defined by a cyclotomic trinomial  $\mathbb{Z}_q[x]/\langle x^n - x^{n/2} + 1 \rangle$ . NTTRU was based on the Dent [14] transformation without any encoding method, which resulted in an approximate worst-case correctness error of  $2^{-13}$ , even with an average-case error of  $2^{-1230}$ . To overcome this significant difference, NTTRU was modified to reduce the message space of the underlying scheme, while increasing the size of the ciphertext. This modification was later generalized to ACWC<sub>0</sub> in [16].

In 2021, Duman et al. [16] proposed two generic transformations, ACWC<sub>0</sub> and ACWC, which aim to make the average-case correctness error of an underlying scheme nearly equal to the worst-case error of the transformed scheme. Specifically, ACWC introduced GOTP as an encoding method to prevent  $\mathcal{A}$  from adversarially choosing  $m$ . While ACWC<sub>0</sub> is simple, it requires a ciphertext expansion of 32 bytes. On the other hand, ACWC does not require an expansion of the ciphertext size. The security of ACWC<sub>0</sub> and ACWC was analyzed in both the classical and quantum random oracle models [16]. However, their NTRU instantiation using ACWC has the drawback of requiring the message  $m$  to be chosen from a uniformly random distribution over  $\mathcal{M}' = \{-1, 0, 1\}^\lambda$ .

## 2 Preliminaries

### 2.1 Basic Notations

The set  $\mathbb{Z}_q$  is defined as  $\{-(q-1)/2, \dots, (q-1)/2\}$ , where  $q$  is a positive odd integer. Mapping an integer  $a$  from  $\mathbb{Z}$  to  $\mathbb{Z}_q$  uses the modulo operation, setting  $x = a \bmod q$  as the unique integer in  $\mathbb{Z}_q$  satisfying  $q \mid (x - a)$ . The polynomial ring  $R_q$  is defined as  $\mathbb{Z}_q[x]/\langle f(x) \rangle$  with a polynomial  $f(x)$ . Cyclotomic trinomials  $\Phi_{3n}(x) = x^n - x^{n/2} + 1$  where  $n = 2^i \cdot 3^j$  for some positive integers  $i$  and  $j$  are used as  $f(x)$  in our construction. Polynomials in  $R_q$  are denoted in non-italic bold as  $\mathbf{a}$ , with  $\mathbf{a}_i$  as the  $i$ -th coefficient.

For sampling,  $u \leftarrow X$  indicates that  $u$  is sampled uniformly at random from a set  $X$ , and  $u \leftarrow D$  indicates that  $u$  is drawn according to a distribution  $D$ . The notation  $u \leftarrow D^\ell$  forms a vector  $u = (u_1, \dots, u_\ell)$  with each  $u_i$  drawn independently from  $D$ . Especially,  $\mathbf{a} \leftarrow D$  indicates that all coefficients of a polynomial  $\mathbf{a}$  are drawn according to a distribution  $D$ . Sampling from the centered binomial distribution (CBD)  $\psi_k$  involves  $2k$  bits that are independent and uniformly random, summing the first  $k$  bits and the second  $k$  bits separately, then outputting their difference.

### 2.2 Public-Key Encryption

**Definition 2.1** (Public Key Encryption). A public-key encryption scheme  $\text{PKE} = (\text{Gen}, \text{Enc}, \text{Dec})$  with message space  $\mathcal{M}$ , randomness space  $\mathcal{R}$ , and ciphertext space  $\mathcal{C}$  consists of the following three algorithms:

- $\text{Gen}(1^\lambda)$ : The key generation algorithm  $\text{Gen}$  is a randomized algorithm that takes a security parameter  $1^\lambda$  as input and outputs a pair of public/secret keys  $(pk, sk)$ .
- $\text{Enc}(pk, m; r)$ : The encryption algorithm  $\text{Enc}$  is a randomized algorithm that takes a public key  $pk$ , a message  $m \in \mathcal{M}$ , and randomness  $r \in \mathcal{R}$  as input and outputs a ciphertext  $c \in \mathcal{C}$ . We often write  $\text{Enc}(pk, m)$  to denote the encryption algorithm without explicitly mentioning the randomness.
- $\text{Dec}(sk, c)$ : The decryption algorithm  $\text{Dec}$  is a deterministic algorithm that takes a secret key  $sk$  and a ciphertext  $c \in \mathcal{C}$  as input and outputs either a message  $m \in \mathcal{M}$  or a special symbol  $\perp \notin \mathcal{M}$  to indicate that  $c$  is not a valid ciphertext.

**Correctness.** We say that  $\text{PKE}$  has a (worst-case) correctness error  $\delta$  [20] if

$$\mathbb{E} \left[ \max_{m \in \mathcal{M}} \Pr[\text{Dec}(sk, \text{Enc}(pk, m)) \neq m] \right] \leq \delta,$$

where the expectation is taken over  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$  and the choice of the random oracles involved (if any). We say that  $\text{PKE}$  has an average-case correctness error  $\delta$  relative to the distribution  $\psi_{\mathcal{M}}$  over  $\mathcal{M}$  if

$$\mathbb{E} [\Pr[\text{Dec}(sk, \text{Enc}(pk, m)) \neq m]] \leq \delta,$$

where the expectation is taken over  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$ , the choice of the random oracles involved (if any), and  $m \leftarrow \psi_{\mathcal{M}}$ .

**Injectivity.** Injectivity of  $\text{PKE}$  is defined via the following  $\text{GAME INJ}$ , which is shown in Figure 2, and the relevant advantage of adversary  $\mathcal{A}$  is

$$\text{Adv}_{\text{PKE}}^{\text{INJ}}(\mathcal{A}) = \Pr[\text{INJ}_{\text{PKE}}^{\mathcal{A}} \Rightarrow 1].$$

Unlike the definition of injectivity in [8, 20], we define the injectivity in a computationally-secure sense.

**GAME INJ**

- 1:  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$
- 2:  $(m, m', r, r') \leftarrow \mathcal{A}(pk)$
- 3:  $c = \text{Enc}(pk, m; r)$
- 4:  $c' = \text{Enc}(pk, m'; r')$
- 5: **return**  $\llbracket (m, m', r, r') \in \mathcal{M}^2 \times \mathcal{R}^2 \wedge (m, r) \neq (m', r') \wedge c = c' \rrbracket$

Figure 2: GAME INJ for PKE

**Spreadness.** PKE is  $\gamma$ -spread [20] if

$$\min_{m \in \mathcal{M}, (sk, pk)} \left( -\log \max_{c \in \mathcal{C}} \Pr_{r \leftarrow \psi_{\mathcal{R}}} [c = \text{Enc}(pk, m; r)] \right) \geq \gamma,$$

where the minimum is taken over all key pairs that can be generated by Gen. This definition can be relaxed by considering an expectation over the choice of  $(pk, sk)$ . PKE is weakly  $\gamma$ -spread [15] if

$$-\log \mathbb{E} \left[ \max_{m \in \mathcal{M}, c \in \mathcal{C}} \Pr_{r \leftarrow \psi_{\mathcal{R}}} [c = \text{Enc}(pk, m; r)] \right] \geq \gamma,$$

where the expectation is over  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$ .

**Randomness-Recoverability.** PKE is defined as *randomness-recoverable* (RR) if there exists an algorithm  $\text{RRec}$  such that, for all key pairs  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$ , and for any message  $m \in \mathcal{M}$  and for any ciphertext  $c \in \mathcal{C}$ , the following condition holds:

$$\text{if } r = \text{RRec}(pk, m, c) \in \mathcal{R}, \text{ then } c = \text{Enc}(pk, m; r).$$

**Arbitrary-Ciphertext Message-Recoverability.** PKE is *arbitrary-ciphertext message-recoverable* (AC-MR) if there exists an algorithm  $\text{MRec}$  such that, for all key pairs  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$ , and for any randomness  $r \in \mathcal{R}$ , and for any ciphertexts  $c \in \mathcal{C}$ ,

$$\text{if } m = \text{MRec}(pk, r, c) \in \mathcal{M}, \text{ then } c = \text{Enc}(pk, m; r).$$

**Valid-Ciphertext Message-Recoverability.** PKE is *valid-ciphertext message-recoverable* (VC-MR) if there exists an algorithm  $\text{MRec}$  such that, for all key pairs  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$ , for any message  $m \in \mathcal{M}$ , for any randomness  $r \in \mathcal{R}$ , and for any ciphertexts  $c \in \mathcal{C}$ ,

$$\text{if } c = \text{Enc}(pk, m; r), \text{ then } \text{MRec}(pk, r, c) = m.$$

**Randomness-Uniqueness.** PKE is defined as *randomness-unique* (RU) if for all key pairs  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$ , and for any message  $m \in \mathcal{M}$  and any randomness  $r, r' \in \mathcal{R}$ , the following condition holds:

$$\text{if } \text{Enc}(pk, m; r) = \text{Enc}(pk, m; r'), \text{ then } r = r'.$$

GAME OW-CPA

- 1:  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$
- 2:  $m \leftarrow \psi_{\mathcal{M}}$
- 3:  $c^* \leftarrow \text{Enc}(pk, m)$
- 4:  $m' \leftarrow \mathcal{A}(pk, c^*)$
- 5: **return**  $\llbracket m = m' \rrbracket$

GAME IND-CPA

- 1:  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$
- 2:  $(m_0, m_1) \leftarrow \mathcal{A}_0(pk)$
- 3:  $b \leftarrow \{0, 1\}$
- 4:  $c^* \leftarrow \text{Enc}(pk, m_b)$
- 5:  $b' \leftarrow \mathcal{A}_1(pk, c^*)$
- 6: **return**  $\llbracket b = b' \rrbracket$

Figure 3: GAMES OW-CPA and IND-CPA for PKE

GAME IND-CCA

- 1:  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$
- 2:  $(K_0, c^*) \leftarrow \text{Encap}(pk)$
- 3:  $K_1 \leftarrow \mathcal{K}$
- 4:  $b \leftarrow \{0, 1\}$
- 5:  $b' \leftarrow \mathcal{A}^{\text{Decap}}(pk, c^*, K_b)$
- 6: **return**  $\llbracket b = b' \rrbracket$

Decap( $c \neq c^*$ )

- 1: **return**  $\text{Decap}(sk, c)$

Figure 4: GAME IND-CCA for KEM

**Definition 2.2** (OW-CPA security of PKE). Let  $\text{PKE} = (\text{Gen}, \text{Enc}, \text{Dec})$  be a public-key encryption scheme with message space  $\mathcal{M}$ . Onewayness under chosen-plaintext attacks (OW-CPA) for message distribution  $\psi_{\mathcal{M}}$  is defined via the GAME OW-CPA, as shown in Figure 3, and the advantage function of adversary  $\mathcal{A}$  is

$$\text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{A}) := \Pr \left[ \text{OW-CPA}_{\text{PKE}}^{\mathcal{A}} \Rightarrow 1 \right].$$

**Definition 2.3** (IND-CPA security of PKE). Let  $\text{PKE} = (\text{Gen}, \text{Enc}, \text{Dec})$  be a public-key encryption scheme with message space  $\mathcal{M}$ . Indistinguishability under chosen-plaintext attacks (IND-CPA) is defined via the GAME IND-CPA, as shown in Figure 3, and the advantage function of adversary  $\mathcal{A}$  is

$$\text{Adv}_{\text{PKE}}^{\text{IND-CPA}}(\mathcal{A}) := \left| \Pr \left[ \text{IND-CPA}_{\text{PKE}}^{\mathcal{A}} \Rightarrow 1 \right] - \frac{1}{2} \right|.$$

### 2.3 Key Encapsulation Mechanism

**Definition 2.4** (Key Encapsulation Mechanism). A key encapsulation mechanism  $\text{KEM} = (\text{Gen}, \text{Encap}, \text{Decap})$  with a key space  $\mathcal{K}$  consists of the following three algorithms:

- $\text{Gen}(1^\lambda)$ : The key generation algorithm  $\text{Gen}$  is a randomized algorithm that takes a security parameter  $\lambda$  as input and outputs a pair of public key and secret key,  $(pk, sk)$ .
- $\text{Encap}(pk)$ : The encapsulation algorithm  $\text{Encap}$  is a randomized algorithm that takes a public key  $pk$  as input, and outputs a ciphertext  $c$  and a key  $K \in \mathcal{K}$ .
- $\text{Decap}(sk, c)$ : The decryption algorithm  $\text{Decap}$  is a deterministic algorithm that takes a secret key  $sk$  and ciphertext  $c$  as input, and outputs either a key  $K \in \mathcal{K}$  or a special symbol  $\perp \notin \mathcal{K}$  to indicate that  $c$  is not a valid ciphertext.

**Correctness.** We say that KEM has a correctness error  $\delta$  if

$$\Pr[\text{Decap}(sk, c) \neq K | (c, K) \leftarrow \text{Encap}(pk)] \leq \delta,$$

where the probability is taken over the randomness in  $\text{Encap}$  and  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$ .

**Definition 2.5** (IND-CCA security of KEM). Let  $\text{KEM} = (\text{Gen}, \text{Encap}, \text{Decap})$  be a key encapsulation mechanism with a key space  $\mathcal{K}$ . Indistinguishability under chosen-ciphertext attacks (IND-CCA) is defined via the GAME IND-CCA, as shown in Figure 4, and the advantage function of adversary  $\mathcal{A}$  is as follows:

$$\text{Adv}_{\text{KEM}}^{\text{IND-CCA}}(\mathcal{A}) := \left| \Pr[\text{IND-CCA}_{\text{KEM}}^{\mathcal{A}} \Rightarrow 1] - \frac{1}{2} \right|.$$

## 2.4 Complexity Assumptions

This section outlines complexity assumptions used in NTRU+. Specifically, it introduces the NTRU and RLWE problems. Unlike the RLWE problem used in ElGamal-type schemes [2], RLWE here is defined in the computational sense.

**Definition 2.6** (The NTRU problem [19]). Let  $\psi$  be a distribution over  $R_q$ . The NTRU problem  $\text{NTRU}_{n,q,\psi}$  is to distinguish  $\mathbf{h} = \mathbf{g}(p\mathbf{f}' + 1)^{-1} \in R_q$  from  $\mathbf{u} \in R_q$ , where  $\mathbf{f}' \leftarrow \psi$  and  $\mathbf{u} \leftarrow R_q$ . The advantage of adversary  $\mathcal{A}$  in solving  $\text{NTRU}_{n,q,\psi}$  is defined as follows:

$$\text{Adv}_{n,q,\psi}^{\text{NTRU}}(\mathcal{A}) = \Pr[\mathcal{A}(\mathbf{h}) = 1] - \Pr[\mathcal{A}(\mathbf{u}) = 1].$$

**Definition 2.7** (The RLWE problem [30]). Let  $\psi$  be a distribution over  $R_q$ . The RLWE problem  $\text{RLWE}_{n,q,\psi}$  is to find  $\mathbf{s}$  from  $(\mathbf{a}, \mathbf{b} = \mathbf{a}\mathbf{s} + \mathbf{e}) \in R_q \times R_q$ , where  $\mathbf{a} \leftarrow R_q$ ,  $\mathbf{s}, \mathbf{e} \leftarrow \psi$ . The advantage of an adversary  $\mathcal{A}$  in solving  $\text{RLWE}_{n,q,\psi}$  is defined as follows:

$$\text{Adv}_{n,q,\psi}^{\text{RLWE}}(\mathcal{A}) = \Pr[\mathcal{A}(\mathbf{a}, \mathbf{b}) = \mathbf{s}].$$

## 2.5 Auxiliary Lemmas for the Security Proofs

**Lemma 2.8** (Fundamental lemma of game-playing [6, Lemma 1]). Let  $G$  and  $H$  be identical-until-bad games, meaning that both games maintain a boolean flag **bad** initially set to false and behave identically until **bad** is set to true. Then, for any adversary  $\mathcal{A}$ ,

$$\left| \Pr[G^{\mathcal{A}} \Rightarrow 1] - \Pr[H^{\mathcal{A}} \Rightarrow 1] \right| \leq \Pr[G^{\mathcal{A}} \text{ sets } \mathbf{bad}].$$

**Lemma 2.9** (Classical O2H, Theorem 3 from the eprint version of [3]). Let  $S \subset \mathcal{R}$  be random. Let  $G$  and  $F$  be random functions satisfying  $\forall r \notin S : G(r) = F(r)$ . Let  $z$  be a random classical value ( $S, G, F, z$  may have an arbitrary joint distribution). Let  $\mathcal{C}$  be a quantum oracle algorithm with query depth  $q_G$ , expecting input  $z$ . Let  $\mathcal{D}$  be the algorithm that, on input  $z$ , samples a uniform  $i$  from  $\{1, \dots, q_G\}$ , runs  $\mathcal{C}$  right before its  $i$ -th query to  $F$ , measures all query input registers, and outputs the set  $T$  of measurement outcomes. Then

$$\left| \Pr[\mathcal{C}^G(z) \Rightarrow 1] - \Pr[\mathcal{C}^F(z) \Rightarrow 1] \right| \leq 2q_G \sqrt{\Pr[S \cap T \neq \emptyset : T \leftarrow \mathcal{D}^F(z)]}.$$

**Lemma 2.10** (Generic search problem [4, 25, 26]). Let  $\gamma \in [0, 1]$ . Let  $Z$  be a finite set.  $N_1 : Z \rightarrow \{0, 1\}$  is the following function: For each  $z$ ,  $N_1(z) = 1$  with probability  $p_z$  ( $p_z \leq \gamma$ ), and  $N_1(z) = 0$  else. Let  $N_2$  be the function with  $\forall z : N_2(z) = 0$ . If an oracle algorithm  $\mathcal{A}$  makes at most  $q$  quantum queries to  $N_1$  (or  $N_2$ ), then

$$\left| \Pr[b = 1 : b \leftarrow \mathcal{A}^{N_1}] - \Pr[b = 1 : b \leftarrow \mathcal{A}^{N_2}] \right| \leq 2q\sqrt{\gamma}.$$

### 3 ACWC<sub>2</sub> Transformation

We introduce our new ACWC transformation, denoted ACWC<sub>2</sub>, by presenting ACWC<sub>2</sub>[PKE, SOTP, G] for a hash function G, as shown in Figure 5. Let PKE' = ACWC<sub>2</sub>[PKE, SOTP, G] be the resulting encryption scheme. By applying ACWC<sub>2</sub> to an underlying PKE, we show that (1) PKE' achieves a worst-case correctness error that is essentially as small as the average-case error of PKE, and (2) PKE' attains tight IND-CPA security provided that PKE is OW-CPA secure.

#### 3.1 SOTP

**Definition 3.1.** A semi-generalized one-time pad SOTP = (Encode, Decode), with a message space  $\mathcal{X}$ , a randomness space  $\mathcal{U}$  (with corresponding distribution  $\psi_{\mathcal{U}}$ ), and a code space  $\mathcal{Y}$  (with corresponding distribution  $\psi_{\mathcal{Y}}$ ), consists of the following two algorithms:

- $\text{Encode}(x, u)$  : The encoding algorithm Encode is a deterministic algorithm that takes a message  $x \in \mathcal{X}$  and randomness  $u \in \mathcal{U}$  as input, and outputs a code  $y \in \mathcal{Y}$ .
- $\text{Decode}(y, u)$  : The decoding algorithm Decode is a deterministic algorithm that takes a code  $y \in \mathcal{Y}$  and randomness  $u \in \mathcal{U}$  as input, and outputs a message  $x \in \mathcal{X} \cup \{\perp\}$ .

It also satisfies the following three properties:

1. Decoding: For all  $x \in \mathcal{X}$  and  $u \in \mathcal{U}$ ,  $\text{Decode}(\text{Encode}(x, u), u) = x$ .
2. Message-hiding: For all  $x \in \mathcal{X}$ ,  $\text{Encode}(x, u)$  with  $u \leftarrow \psi_{\mathcal{U}}$  follows the distribution  $\psi_{\mathcal{Y}}$ .
3. Rigid: For all  $u \in \mathcal{U}$  and  $y \in \mathcal{Y}$  such that  $\text{Decode}(y, u) \neq \perp$ , we have  $\text{Encode}(\text{Decode}(y, u), u) = y$ .

In contrast to the GOTP defined in [16], SOTP does not need to satisfy an additional *randomness-hiding* property, which requires that the output  $y = \text{Encode}(x, u)$  follow the distribution  $\psi_{\mathcal{Y}}$  while simultaneously revealing no information about the randomness  $u$ . The absence of this additional requirement allows SOTP to be designed more flexibly and efficiently than GOTP. Instead, SOTP is required to be *rigid*, meaning that for all  $u \in \mathcal{U}$  and  $y \in \mathcal{Y}$ , if  $x = \text{Decode}(y, u) \neq \perp$ , then  $\text{Encode}(x, u) = y$ .

#### 3.2 ACWC<sub>2</sub>

Let PKE = (Gen, Enc, Dec) be an underlying public-key encryption scheme with message space  $\mathcal{M}$  and randomness space  $\mathcal{R}$ , where messages  $M \in \mathcal{M}$  and randomness  $r \in \mathcal{R}$  are drawn from the distributions  $\psi_{\mathcal{M}}$  and  $\psi_{\mathcal{R}}$ , respectively. Similarly, let PKE' = (Gen', Enc', Dec') be the transformed encryption scheme with message space  $\mathcal{M}'$  and randomness space  $\mathcal{R}'$ . Let SOTP = (Encode, Decode) be a semi-generalized one-time pad, where  $\text{Encode} : \mathcal{M}' \times \mathcal{U} \rightarrow \mathcal{M}$  and  $\text{Decode} : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}'$  are defined with respect to distributions  $\psi_{\mathcal{U}}$  and  $\psi_{\mathcal{M}}$ . Let  $G : \mathcal{R} \rightarrow \mathcal{U}$  be a hash function whose outputs are independently  $\psi_{\mathcal{U}}$ -distributed. Then PKE' = ACWC<sub>2</sub>[PKE, SOTP, G] is described in Figure 5.

Under the condition that  $\text{Dec}(sk, c)$  in Dec' yields the same  $M$  as in Enc, the deterministic functions RRec and Decode do not affect the correctness error of PKE'. Thus, the factor that determines the success or failure of  $\text{Dec}'(sk, c)$  is the result of  $\text{Dec}(sk, c)$  within Dec'. This implies that the correctness error of PKE is directly transferred to that of PKE', and is eventually determined by how the randomness  $r \in \mathcal{R}$  and message  $M \in \mathcal{M}$  are sampled in PKE'. We observe that  $r$  is drawn according to the distribution  $\psi_{\mathcal{R}}$  and that  $M$  is

| <u>Gen'(1<math>^\lambda</math>)</u>   | <u>Dec'(sk, c)</u>   |
|---|--|
| 1: $(pk, sk) := \text{Gen}(1^\lambda)$  | 1: $M := \text{Dec}(sk, c)$  |
| 2: <b>return</b> $(pk, sk)$   | 2: $r := \text{RRec}(pk, M, c)$  |
| <u>Enc'(pk, m <math>\in \mathcal{M}'</math>; R <math>\in \mathcal{R}'</math>)</u> | 3: $m := \text{Decode}(M, G(r))$   |
| 1: $r \leftarrow \psi_{\mathcal{R}}$ using the randomness R                       | 4: <b>if</b> $r \notin \mathcal{R}$ or $m = \perp$ , <b>return</b> $\perp$ |
| 2: $M := \text{Encode}(m, G(r))$  | 5: <b>return</b> $m$   |
| 3: $c := \text{Enc}(pk, M; r)$  |  |
| 4: <b>return</b> $c$  |  |

Figure 5: ACWC<sub>2</sub>[PKE, SOTP, G]

an SOTP-encoded element in  $\mathcal{M}$ . Because each output of  $G$  is independently  $\psi_{\mathcal{U}}$ -distributed, the message-hiding property of SOTP ensures that  $M$  follows the distribution  $\psi_{\mathcal{M}}$  while hiding  $m$ . Consequently, both  $M$  and  $r$  are chosen according to their originally intended distributions.

However, since the choice of the random oracle  $G$  can affect the correctness error of PKE', we need to incorporate this observation into the correctness analysis. Theorem 3.2 shows that, for all but a negligible fraction of random oracles  $G$ , the worst-case correctness of PKE' (transformed by ACWC<sub>2</sub>) is close to the average-case correctness of PKE. This mirrors the idea underlying ACWC, and the proof strategy of Theorem 3.2 is essentially the same as in [16] (Lemma 3.6 therein), except for slight modifications to the message distribution.

**Theorem 3.2** (Average-Case to Worst-Case Correctness error). Let PKE be RR and have a randomness space  $\mathcal{R}$  relative to the distribution  $\psi_{\mathcal{R}}$ . Let SOTP = (Encode, Decode), with  $\text{Encode} : \mathcal{M}' \times \mathcal{U} \rightarrow \mathcal{M}$  and  $\text{Decode} : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}'$ , be a semi-generalized one-time pad (for distributions  $\psi_{\mathcal{U}}$  and  $\psi_{\mathcal{M}}$ ). Let  $G : \mathcal{R} \rightarrow \mathcal{U}$  be a random oracle whose outputs are distributed according to  $\psi_{\mathcal{U}}$ . If PKE is  $\delta$ -average-case-correct, then PKE' := ACWC<sub>2</sub>[PKE, SOTP, G] is  $\delta'$ -worst-case-correct for

$$\delta' = \delta + \|\psi_{\mathcal{R}}\| \cdot \left( 1 + \sqrt{(\ln |\mathcal{M}'| - \ln \|\psi_{\mathcal{R}}\|)/2} \right),$$

where  $\|\psi_{\mathcal{R}}\| := \sqrt{\sum_r \psi_{\mathcal{R}}(r)^2}$ .

*Proof.* Taking expectation over  $G$  and  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$ , the worst-case correctness of PKE' is

$$\delta' = \mathbb{E} \left[ \max_{m \in \mathcal{M}'} \Pr_{G} [\text{Dec}'(sk, \text{Enc}'(pk, m)) \neq m] \right] = \mathbb{E}[\delta'(pk, sk)],$$

where  $\delta'(pk, sk) := \mathbb{E}[\max_{m \in \mathcal{M}'} \Pr[\text{Dec}'(sk, \text{Enc}'(pk, m)) \neq m]]$  is the expectation over  $G$  for a fixed key pair  $(pk, sk)$ . For any fixed key pair and any positive real  $t \in \mathbb{R}^+$ , we have

$$\begin{aligned} \delta'(pk, sk) &= \mathbb{E} \left[ \max_{m \in \mathcal{M}'} \Pr_{G} [\text{Dec}'(sk, \text{Enc}'(pk, m)) \neq m] \right] \\ &\leq t + \Pr_{G} \left[ \max_{m \in \mathcal{M}'} \Pr_{G} [\text{Dec}'(sk, \text{Enc}'(pk, m)) \neq m] \geq t \right] \end{aligned} \tag{1}$$

$$\begin{aligned} &\leq t + \Pr_{G} \left[ \max_{m \in \mathcal{M}'} \Pr_{r \leftarrow \psi_{\mathcal{R}}} [\text{Dec}'(sk, \text{Enc}(pk, M; r)) \neq m] \geq t \right] \\ &\leq t + \Pr_{G} \left[ \max_{m \in \mathcal{M}'} \Pr_{r \leftarrow \psi_{\mathcal{R}}} [\text{Dec}(sk, \text{Enc}(pk, M; r)) \neq M] \geq t \right], \end{aligned} \tag{2}$$

where  $M = \text{Encode}(m, G(r))$ . Note that Equation (1) holds by Lemma 3.4, and Equation (2) holds because

$$\Pr_{r \leftarrow \psi_{\mathcal{R}}} [\text{Dec}'(sk, \text{Enc}(pk, M; r)) \neq m] \leq \Pr_{r \leftarrow \psi_{\mathcal{R}}} [\text{Dec}(sk, \text{Enc}(pk, M; r)) \neq M].$$

For any fixed key pair and any positive  $c$ , let  $t(pk, sk) := \mu(pk, sk) + \|\psi_{\mathcal{R}}\| \cdot \sqrt{(c + \ln |\mathcal{M}'|)/2}$ , where  $\mu(pk, sk) := \Pr_{M, r} [\text{Dec}(sk, \text{Enc}(pk, M; r)) \neq M]$ . Then, we can use Lemma 3.5 to argue that

$$\Pr_{\mathcal{G}} \left[ \max_{m \in \mathcal{M}'} \Pr_{r \leftarrow \psi_{\mathcal{R}}} [\text{Dec}(sk, \text{Enc}(pk, M; r)) \neq M] > t(pk, sk) \right] \leq e^{-c}. \quad (3)$$

To this end, we define

$$g(m, r, u) = (\text{Encode}(m, u), r) \quad \text{and} \quad B = \{(M, r) \in \mathcal{M} \times \mathcal{R} \mid \text{Dec}(sk, \text{Enc}(pk, M; r)) \neq M\},$$

which will be used in Lemma 3.5. Note that  $\Pr_{r \leftarrow \psi_{\mathcal{R}}, u \leftarrow \psi_{\mathcal{U}}} [g(m, r, u) \in B] = \mu(pk, sk)$  holds for all  $m \in \mathcal{M}'$  by the message-hiding property of the SOTP. For all  $m \in \mathcal{M}'$ ,

$$\begin{aligned} \Pr_{r \leftarrow \psi_{\mathcal{R}}, u \leftarrow \psi_{\mathcal{U}}} [g(m, r, u) \in B] &= \Pr_{r \leftarrow \psi_{\mathcal{R}}, u \leftarrow \psi_{\mathcal{U}}} [(\text{Encode}(m, u), r) \in B] \\ &= \Pr_{r \leftarrow \psi_{\mathcal{R}}, M \leftarrow \psi_{\mathcal{M}}} [(M, r) \in B] \\ &= \Pr_{r \leftarrow \psi_{\mathcal{R}}, M \leftarrow \psi_{\mathcal{M}}} [\text{Dec}(sk, \text{Enc}(pk, M; r)) \neq M] \\ &= \mu(pk, sk). \end{aligned}$$

Combining Equation (3) with Equation (2) and taking the expectation yields

$$\begin{aligned} \delta' &\leq \mathbb{E} \left[ \mu(pk, sk) + \|\psi_{\mathcal{R}}\| \cdot \sqrt{(c + \ln |\mathcal{M}'|)/2} + e^{-c} \right] \\ &= \delta + \|\psi_{\mathcal{R}}\| \cdot \sqrt{(c + \ln |\mathcal{M}'|)/2} + e^{-c}, \end{aligned}$$

and setting  $c := -\ln \|\psi_{\mathcal{R}}\|$  yields the claim in the theorem.  $\square$

**Corollary 3.3** (Average-Case to Worst-Case Correctness error). Let PKE be RR and have a randomness space  $\mathcal{R}$  relative to the distribution  $\psi_{\mathcal{R}}$ . Let SOTP = ( $\text{Encode}$ ,  $\text{Decode}$ ), with  $\text{Encode} : \mathcal{M}' \times \mathcal{U} \rightarrow \mathcal{M}$  and  $\text{Decode} : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}'$ , be a semi-generalized one-time pad (for distributions  $\psi_{\mathcal{U}}$  and  $\psi_{\mathcal{M}}$ ). Let  $G : \mathcal{R} \rightarrow \mathcal{U}$  be a random oracle whose outputs are distributed according to  $\psi_{\mathcal{U}}$ . If PKE is  $\delta$ -average-case-correct, then

$$\mathbb{E} \left[ \max_{m \in \mathcal{M}'} \Pr_{r \leftarrow \psi_{\mathcal{R}}} [\text{Dec}(sk, \text{Enc}(pk, M; r)) \neq M] \right] \leq \delta',$$

where  $M = \text{Encode}(m, G(r))$ , and

$$\delta' = \delta + \|\psi_{\mathcal{R}}\| \cdot \left( 1 + \sqrt{(\ln |\mathcal{M}'| - \ln \|\psi_{\mathcal{R}}\|)/2} \right),$$

with  $\|\psi_{\mathcal{R}}\| := \sqrt{\sum_r \psi_{\mathcal{R}}(r)^2}$ , where the expectation is over  $G$  and  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$ .

*Proof.* The proof is analogous to that of Theorem 3.2, and we omit the detailed argument.  $\square$

**Lemma 3.4.** Let  $X$  be a random variable and let  $f$  be a non-negative real-valued function with  $f(X) \leq 1$ . Then, for all  $t \in \mathbb{R}^+$ ,

$$\mathbb{E}[f(X)] \leq t + \Pr[f(X) \geq t].$$

*Proof.* By applying the law of total probability and by partitioning the domain of  $x$  into the cases where  $f(x) < t$  and  $f(x) \geq t$ , we obtain

$$\begin{aligned} \mathbb{E}[f(X)] &= \sum_x f(x) \Pr[X = x] \\ &= \sum_{f(x) < t} f(x) \Pr[X = x] + \sum_{f(x) \geq t} f(x) \Pr[X = x] \\ &\leq \sum_{f(x) < t} t \Pr[X = x] + \sum_{f(x) \geq t} f(x) \Pr[X = x] \\ &\leq t + \sum_{f(x) \geq t} f(x) \Pr[X = x] \\ &\leq t + \sum_{f(x) \geq t} \Pr[X = x] = t + \Pr[f(X) \geq t]. \end{aligned}$$

The final equality follows from  $\sum_{f(x) \geq t} \Pr[X = x] = \Pr[f(X) \geq t]$ .  $\square$

**Lemma 3.5** (Adapting Lemma 3.7 from [16]). Let  $g$  be a function, and let  $B$  be a set such that

$$\forall m \in \mathcal{M}', \quad \Pr_{r \leftarrow \psi_{\mathcal{R}}, u \leftarrow \psi_{\mathcal{U}}} [g(m, r, u) \in B] \leq \mu \quad (4)$$

for some  $\mu \in [0, 1]$ . Let  $\mathsf{G} : \mathcal{R} \rightarrow \mathcal{U}$  be a random function whose outputs are independently distributed according to  $\psi_{\mathcal{U}}$ . Define  $\|\psi_{\mathcal{R}}\| = \sqrt{\sum_r \psi_{\mathcal{R}}(r)^2}$ . Then, for all but an  $e^{-c}$  fraction of random functions  $\mathsf{G}$ , we have that  $\forall m \in \mathcal{M}'$ ,

$$\Pr_{r \leftarrow \psi_{\mathcal{R}}} [g(m, r, \mathsf{G}(r)) \in B] \leq \mu + \|\psi_{\mathcal{R}}\| \cdot \sqrt{(c + \ln |\mathcal{M}'|)/2} \quad (5)$$

for any  $c \in \mathbb{R}^+$ .

*Proof.* For fixed  $m \in \mathcal{M}'$  and  $r \in \mathcal{R}$ , define  $p_r := \Pr_{u \leftarrow \psi_{\mathcal{U}}} [g(m, r, u) \in B]$ . From Equation (4), we know that  $\sum_r \psi_{\mathcal{R}}(r) p_r \leq \mu$ . For each  $r$ , define a random variable  $X_r$  whose value is determined as follows:  $\mathsf{G}$  chooses a random  $u = \mathsf{G}(r)$  and then checks whether  $g(m, r, \mathsf{G}(r)) \in B$ ; if it does, then we set  $X_r = 1$ ; otherwise we set it to zero. Because  $\mathsf{G}$  is a random function, the probability that  $X_r = 1$  is exactly  $p_r$ .

The probability of Equation (5) for our particular  $m$  is the same as the sum  $\sum_r \psi_{\mathcal{R}}(r) X_r$ , and we use the Hoeffding bound to show that this value is not significantly larger than  $\mu$ . We define the random variable  $Y_r = \psi_{\mathcal{R}}(r) X_r$ . Notice that  $Y_r \in [0, \psi_{\mathcal{R}}(r)]$ , and  $\mathbb{E}[\sum_r Y_r] = \mathbb{E}[\sum_r \psi_{\mathcal{R}}(r) X_r] = \sum_r \psi_{\mathcal{R}}(r) p_r \leq \mu$ . By the Hoeffding bound, we have for all positive  $t$ ,

$$\Pr_r [\sum_r Y_r > \mu + t] \leq \exp \left( \frac{-2t^2}{\sum_r \psi_{\mathcal{R}}(r)^2} \right) = \exp \left( \frac{-2t^2}{\|\psi_{\mathcal{R}}\|^2} \right). \quad (6)$$

By setting  $t \geq \|\psi_{\mathcal{R}}\| \cdot \sqrt{(c + \ln |\mathcal{M}'|)/2}$ , for a fixed  $m$ , Equation (5) holds for all but an  $e^{-c} \cdot |\mathcal{M}'|^{-1}$  fraction of random functions  $\mathsf{G}$ . Applying the union bound yields the claim in the lemma.  $\square$

| GAMES $G_0$ - $G_2$  |                 | $G(r)$  | $//G_1$ - $G_2$ |
|--|-----------------|---|-----------------|
| 1: $\mathsf{G} \leftarrow (\mathcal{R} \rightarrow \mathcal{U})$ | $//G_0$         | 1: <b>if</b> $\exists (r, u) \in \mathcal{L}_{\mathsf{G}}$                |                 |
| 2: $\mathcal{L}_{\mathsf{G}}, \mathcal{L}_r := \emptyset$        | $//G_1$ - $G_2$ | 2: <b>return</b> $u$  |                 |
| 3: $(pk, sk) \leftarrow \mathsf{Gen}(1^\lambda)$                 |                 | 3: $u \leftarrow \psi_{\mathcal{U}}$                                      |                 |
| 4: $(m_0, m_1) \leftarrow \mathcal{A}_0^{\mathsf{G}}(pk)$        |                 | 4: $\mathcal{L}_{\mathsf{G}} := \mathcal{L}_{\mathsf{G}} \cup \{(r, u)\}$ |                 |
| 5: $b \leftarrow \{0, 1\}$                                       |                 | 5: $\mathcal{L}_r := \mathcal{L}_r \cup \{r\}$                            |                 |
| 6: $r^* \leftarrow \psi_{\mathcal{R}}$                           |                 | 6: <b>return</b> $u$  |                 |
| 7: $M^* = \mathsf{Encode}(m_b, \mathsf{G}(r^*))$                 | $//G_1$         |   |                 |
| 8: $M^* \leftarrow \psi_{\mathcal{M}}$                           | $//G_2$         |   |                 |
| 9: $c^* \leftarrow \mathsf{Enc}(pk, M^*; r^*)$                   |                 |   |                 |
| 10: $b' \leftarrow \mathcal{A}_1^{\mathsf{G}}(pk, c^*)$          |                 |   |                 |
| 11: <b>return</b> $\llbracket b = b' \rrbracket$                 |                 |   |                 |

Figure 6: GAMES  $G_0$ - $G_2$  of Theorem 3.6

### 3.2.1 Security Proof in the ROM

**Theorem 3.6** (OW-CPA security of PKE  $\xrightarrow{\text{ROM}}$  IND-CPA security of ACWC<sub>2</sub>[PKE, SOTP, G]). Let PKE be a public-key encryption scheme with AC-MR, VC-MR, and RR properties. For any adversary  $\mathcal{A}$  against the IND-CPA security of ACWC<sub>2</sub>[PKE, SOTP, G], making at most  $q_{\mathsf{G}}$  random oracle queries, there exists an adversary  $\mathcal{B}$  against the injectivity of PKE and an adversary  $\mathcal{C}$  against the OW-CPA security of PKE with

$$\text{Adv}_{\text{ACWC}_2[\text{PKE}, \text{SOTP}, \mathsf{G}]}^{\text{IND-CPA}}(\mathcal{A}) \leq \text{Adv}_{\text{PKE}}^{\text{INJ}}(\mathcal{B}) + \text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{C}),$$

where the running time of  $\mathcal{B}$  and  $\mathcal{C}$  is about  $\text{Time}(\mathcal{A}) + O(q_{\mathsf{G}})$ .

*Proof.* We prove the theorem by constructing adversaries  $\mathcal{B}$  (Figure 7) and  $\mathcal{C}$  (Figure 8) from  $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$ , where  $\mathcal{B}$  and  $\mathcal{C}$  break the injectivity and the OW-CPA security of PKE, respectively, while  $\mathcal{A}$  breaks the IND-CPA security of ACWC<sub>2</sub>[PKE, SOTP, G].

GAME  $G_0$ .  $G_0$  (see Figure 6) is the original IND-CPA game with ACWC<sub>2</sub>[PKE, SOTP, G]. By the definition of the IND-CPA game,

$$|\Pr[G_0^{\mathcal{A}} \Rightarrow 1] - \frac{1}{2}| = \text{Adv}_{\text{ACWC}_2[\text{PKE}, \text{SOTP}, \mathsf{G}]}^{\text{IND-CPA}}(\mathcal{A}).$$

GAME  $G_1$ .  $G_1$  is identical to  $G_0$  except that the random oracle  $\mathsf{G}$  is instantiated via lazy sampling rather than by sampling a full random function in advance. Concretely, we maintain a table  $\mathcal{L}_{\mathsf{G}}$ : upon a query  $r$ , if  $r$  has not yet been assigned a value, a fresh  $u \leftarrow \psi_{\mathcal{U}}$  is sampled and  $(r, u)$  is added to  $\mathcal{L}_{\mathsf{G}}$ ; otherwise the stored value is returned. Also, every input  $r$  queried by  $\mathcal{A}$  is also recorded in the set  $\mathcal{L}_r$ . Since lazy sampling is equivalent to a uniform function, we have

$$\Pr[G_0^{\mathcal{A}} \Rightarrow 1] = \Pr[G_1^{\mathcal{A}} \Rightarrow 1].$$

GAME  $G_2$ .  $G_2$  is identical to  $G_1$  except that  $M^*$  is sampled directly from  $\psi_{\mathcal{M}}$  rather than being computed as  $M^* = \mathsf{Encode}(m_b, \mathsf{G}(r^*))$ . Let  $\text{BAD}_1$  be the event that  $\mathcal{A}$  queries  $\mathsf{G}$  at input  $r^*$  in  $G_2$ . If  $\text{BAD}_1$  does not occur, then  $\mathsf{G}(r^*)$  is uniform and independent of  $\mathcal{A}$ 's view in  $G_1$ , hence by the message-hiding of SOTP the

| $\mathcal{B}(pk)$   | $\mathcal{G}(r)$                                    |
|---|---|
| 1: $\mathcal{L}_G, \mathcal{L}_r := \emptyset$                  |   |
| 2: $(m_0, m_1) \leftarrow \mathcal{A}_0^G(pk)$                  | 1: <b>if</b> $\exists (r, u) \in \mathcal{L}_G$     |
| 3: $b \leftarrow \{0, 1\}$                                      | 2: <b>return</b> $u$                                |
| 4: $(M^*, r^*) \leftarrow \psi_M \times \psi_R$                 | 3: $u \leftarrow \psi_U$                            |
| 5: $c^* \leftarrow \text{Enc}(pk, M^*; r^*)$                    | 4: $\mathcal{L}_G := \mathcal{L}_G \cup \{(r, u)\}$ |
| 6: $b' \leftarrow \mathcal{A}_1^G(pk, c^*)$                     | 5: $\mathcal{L}_r := \mathcal{L}_r \cup \{r\}$      |
| 7: <b>for</b> $r \in \mathcal{L}_r$ with $r \neq r^*$ <b>do</b> | 6: <b>return</b> $u$                                |
| 8: $M := \text{MRec}(pk, r, c^*)$                               |   |
| 9: <b>if</b> $(M, r) \in \mathcal{M} \times \mathcal{R}$        |   |
| 10: <b>return</b> $(M, M^*, r, r^*)$                            |   |
| 11: $(M, r) \leftarrow \psi_M \times \psi_R$                    |   |
| 12: <b>return</b> $(M, r, M^*, r^*)$                            |   |

Figure 7: Adversary  $\mathcal{B}$  for the proof of Theorem 3.6

distribution of  $M^* = \text{Encode}(m_b, \mathcal{G}(r^*))$  equals  $\psi_M$ . Therefore, by Lemma 2.8, unless  $\text{BAD}_1$  occurs the distributions of  $G_1$  and  $G_2$  are identical, and

$$|\Pr[G_1^A \Rightarrow 1] - \Pr[G_2^A \Rightarrow 1]| \leq \Pr[\text{BAD}_1 \text{ in } G_2^A].$$

Moreover, in  $G_2$  the pair  $(M^*, r^*)$  is sampled independently of  $b$ , hence

$$\Pr[G_2^A \Rightarrow 1] = \frac{1}{2}.$$

Assume that  $\text{BAD}_1$  occurs in  $G_2$ . Let  $\text{BAD}_2$  be the event that  $\mathcal{A}$  queries some  $r \in \mathcal{L}_r$  with  $r \neq r^*$  and  $M := \text{MRec}(pk, r, c^*) \in \mathcal{M}$ . Then,

$$\begin{aligned} \Pr[\text{BAD}_1 \text{ in } G_2^A] &= \Pr[\text{BAD}_1 \wedge \text{BAD}_2 \text{ in } G_2^A] + \Pr[\text{BAD}_1 \wedge \neg \text{BAD}_2 \text{ in } G_2^A] \\ &\leq \Pr[\text{BAD}_2 \text{ in } G_2^A] + \Pr[\text{BAD}_1 \wedge \neg \text{BAD}_2 \text{ in } G_2^A]. \end{aligned}$$

Assume that the event  $\text{BAD}_2$  occurs in  $G_2$ . We construct an adversary  $\mathcal{B}$  against GAME INJ for PKE as follows. Given  $pk$  from its challenger,  $\mathcal{B}$  runs  $\mathcal{A}_0^G(pk)$ , samples  $(M^*, r^*) \leftarrow \psi_M \times \psi_R$ , and generates  $c^* \leftarrow \text{Enc}(pk, M^*; r^*)$ . It then runs  $\mathcal{A}_1^G(pk, c^*)$  while simulating the random oracle  $G$ . Since  $\text{BAD}_2$  occurs, there exists some  $r \in \mathcal{L}_r$  with  $r \neq r^*$  such that  $M := \text{MRec}(pk, r, c^*) \in \mathcal{M}$ . By the AC-MR property,  $(M, r)$  is a valid pre-image of  $c^*$ . Hence,  $\mathcal{B}$  can identify such an  $r \neq r^*$  in  $\mathcal{L}_r$ , compute  $M := \text{MRec}(pk, r, c^*)$ , and output  $(M, r, M^*, r^*)$ . Therefore,  $\mathcal{B}$  breaks GAME INJ for PKE whenever  $\text{BAD}_2$  occurs. Thus,

$$\Pr[\text{BAD}_2 \text{ in } G_2] \leq \text{Adv}_{\text{PKE}}^{\text{INJ}}(\mathcal{B}).$$

Assuming that the event  $\text{BAD}_1 \wedge \neg \text{BAD}_2$  occurs in  $G_2^A$ , we construct an adversary  $\mathcal{C}$  against GAME OW-CPA for PKE as follows. The challenger provides  $\mathcal{C}$  with  $(pk, c^*)$ , where  $c^*$  is generated by sampling  $(M^*, r^*) \leftarrow \psi_M \times \psi_R$  and computing  $c^* \leftarrow \text{Enc}(pk, M^*; r^*)$ . Upon receiving  $(pk, c^*)$ , the adversary  $\mathcal{C}$  runs  $b \leftarrow \mathcal{A}_0^G(pk)$  and then invokes  $\mathcal{A}_1^G(pk, c^*)$ , while simulating the random oracle  $G$ . Since we assumed that  $\text{BAD}_1 \wedge \neg \text{BAD}_2$  occurs, we have  $r^* \in \mathcal{L}_r$ , and by the VC-MR property it follows that  $M^* = \text{MRec}(pk, r^*, c^*) \in \mathcal{M}$

$\mathcal{C}(pk, c^*)$ 

- 1:  $\mathcal{L}_G, \mathcal{L}_r := \emptyset$
- 2:  $(m_0, m_1) \leftarrow \mathcal{A}_0^G(pk)$
- 3:  $b' \leftarrow \mathcal{A}_1^G(pk, c^*)$
- 4: **for**  $r \in \mathcal{L}_r$  **do**
- 5:    $M := \text{MRec}(pk, r, c^*)$
- 6:   **if**  $(M, r) \in \mathcal{M} \times \mathcal{R}$
- 7:     **return**  $M$
- 8: **return**  $M \leftarrow \psi_{\mathcal{M}}$

 $\mathcal{G}(r)$ 

- 1: **if**  $\exists (r, u) \in \mathcal{L}_G$
- 2:   **return**  $u$
- 3:  $u \leftarrow \psi_{\mathcal{U}}$
- 4:  $\mathcal{L}_G := \mathcal{L}_G \cup \{(r, u)\}$
- 5:  $\mathcal{L}_r := \mathcal{L}_r \cup \{r\}$
- 6: **return**  $u$

Figure 8: Adversary  $\mathcal{C}$  for the proof of Theorem 3.6

and  $c^* = \text{Enc}(pk, M^*; r^*)$ . Moreover, by the assumption  $\neg \text{BAD}_2$ , there exists no  $r \neq r^*$  such that  $M := \text{MRec}(pk, r, c^*) \in \mathcal{M}$ . Thus,  $\mathcal{C}$  can recover  $M^*$  uniquely. Hence,

$$\Pr[\text{BAD}_1 \wedge \neg \text{BAD}_2 \text{ in } G_2^{\mathcal{A}}] \leq \text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{C}).$$

Putting everything together,

$$\begin{aligned} \text{Adv}_{\text{ACWC}_2[\text{PKE}, \text{SOTP}, G]}^{\text{IND-CPA}}(\mathcal{A}) &= \left| \Pr[G_0^{\mathcal{A}} \Rightarrow 1] - \frac{1}{2} \right| \\ &\leq \sum_{i=0}^1 \left| \Pr[G_i^{\mathcal{A}} \Rightarrow 1] - \Pr[G_{i+1}^{\mathcal{A}} \Rightarrow 1] \right| + \left| \Pr[G_2^{\mathcal{A}} \Rightarrow 1] - \frac{1}{2} \right| \\ &\leq \Pr[\text{BAD}_2 \text{ in } G_2] + \Pr[\text{BAD}_1 \wedge \neg \text{BAD}_2 \text{ in } G_2^{\mathcal{A}}] \\ &\leq \text{Adv}_{\text{PKE}}^{\text{INJ}}(\mathcal{B}) + \text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{C}). \end{aligned}$$

□

### 3.2.2 Security Proof in the QROM

**Theorem 3.7** (OW-CPA security of PKE  $\xrightarrow{\text{QROM}}$  IND-CPA security of  $\text{ACWC}_2[\text{PKE}, \text{SOTP}, G]$ ). Let PKE be a public-key encryption scheme with AC-MR, VC-MR, and RR properties. For any quantum adversary  $\mathcal{A}$  against the IND-CPA security of  $\text{ACWC}_2[\text{PKE}, \text{SOTP}, G]$  with query depth at most  $q_G$ , there exists a quantum adversary  $\mathcal{B}$  against the injectivity of PKE and a quantum adversary  $\mathcal{C}$  against the OW-CPA security of PKE with

$$\text{Adv}_{\text{ACWC}_2[\text{PKE}, \text{SOTP}, G]}^{\text{IND-CPA}}(\mathcal{A}) \leq 2q_G \sqrt{\text{Adv}_{\text{PKE}}^{\text{INJ}}(\mathcal{B}) + \text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{C})},$$

where the running time of  $\mathcal{B}$  and  $\mathcal{C}$  is bounded by  $\text{Time}(\mathcal{A}) + O(q_G)$ .

*Proof.* To prove the theorem, we consider a sequence of games  $G_0$  through  $G_7$ , defined in Figures 9, 10, and 11. We first analyze the transition from  $G_0$  to  $G_2$ , and then apply Lemma 2.9 to bound the hop from  $G_2$  to  $G_3$ . A detailed description of the security proof is given below.

|  |
|--|
| <b>GAME <math>G_0</math></b>                                     |
| 1: $\mathbf{G} \leftarrow (\mathcal{R} \rightarrow \mathcal{U})$ |
| 2: $(pk, sk) \leftarrow \mathbf{Gen}(1^\lambda)$                 |
| 3: $(m_0, m_1) \leftarrow \mathcal{A}_0^{\mathbf{G}}(pk)$        |
| 4: $b \leftarrow \{0, 1\}$                                       |
| 5: $r^* \leftarrow \psi_{\mathcal{R}}$                           |
| 6: $M^* \leftarrow \mathbf{Encode}(m_b, \mathbf{G}(r^*))$        |
| 7: $c^* \leftarrow \mathbf{Enc}(pk, M^*; r^*)$                   |
| 8: $b' \leftarrow \mathcal{A}_1^{\mathbf{G}}(pk, c^*)$           |
| 9: <b>return</b> $\llbracket b = b' \rrbracket$                  |

Figure 9: GAME  $G_0$  for Theorem 3.7

GAME  $G_0$ .  $G_0$  (see Figure 9) is the original IND-CPA game with  $\mathbf{ACWC}_2[\mathbf{PKE}, \mathbf{SOTP}, \mathbf{G}]$ . By definition, we have

$$\left| \Pr[G_0^{\mathcal{A}} \Rightarrow 1] - \frac{1}{2} \right| = \mathbf{Adv}_{\mathbf{ACWC}_2[\mathbf{PKE}, \mathbf{SOTP}, \mathbf{G}]}^{\text{IND-CPA}}(\mathcal{A}).$$

GAME  $G_1$ . We define  $G_1$  in the same way as  $G_0$ , except that part of the challenger's logic is encapsulated into an algorithm  $\mathcal{C}^{\mathbf{G}}$ . In addition, before  $\mathcal{C}^{\mathbf{G}}$  runs adversary  $\mathcal{A}$ , we sample  $r^* \leftarrow \psi_{\mathcal{R}}$  and make a classical query  $u := \mathbf{G}(r^*)$ . As these changes are only conceptual, it follows that

$$\Pr[G_0^{\mathcal{A}} \Rightarrow 1] = \Pr[G_1^{\mathcal{A}} \Rightarrow 1].$$

GAME  $G_2$ . We define  $G_2$  in the same way as  $G_1$ , except that we change the way  $\mathbf{G}$  is defined. Instead of choosing  $\mathbf{G}$  uniformly, we sample  $\mathbf{F}$  and  $u$  uniformly and then set  $\mathbf{G} := \mathbf{F}(r^* := u)$ . In other words,  $\mathbf{G}$  is identical to  $\mathbf{F}$  except that it returns  $u$  on input  $r^*$ . Since the distribution of  $(\mathbf{G}, u)$  remains the same, we have

$$\Pr[G_1^{\mathcal{A}} \Rightarrow 1] = \Pr[G_2^{\mathcal{A}} \Rightarrow 1].$$

GAME  $G_3$ . We define  $G_3$  in the same way as  $G_2$ , except that algorithm  $\mathcal{C}$  has oracle access to  $\mathbf{F}$  instead of  $\mathbf{G}$ . Recall that in  $G_2$  the oracle  $\mathbf{G}$  is defined as  $\mathbf{F}(r^* := u)$ , which coincides with  $\mathbf{F}$  on all inputs except  $r^*$ , i.e., they differ only on the set  $S := \{r^*\}$ . This is exactly the setting of Lemma 2.9 with the set  $S$  and auxiliary input  $z := (r^*, u)$ . Hence, by applying the lemma with algorithm  $\mathcal{C}$ , we obtain

$$\left| \Pr[G_2^{\mathcal{A}} \Rightarrow 1] - \Pr[G_3^{\mathcal{A}} \Rightarrow 1] \right| \leq 2q_{\mathbf{G}} \sqrt{\Pr[G_4 \Rightarrow 1]}.$$

Moreover, since the random value  $u$  is only used in  $\mathbf{Encode}(m_b, u)$ , the message-hiding property of  $\mathbf{SOTP}$  implies that  $M^*$  is independent of  $m_b$ . Hence,

$$\Pr[G_3^{\mathcal{A}} \Rightarrow 1] = \frac{1}{2}.$$

GAME  $G_4$  and  $G_5$ . We define  $G_4$  in the same way as  $G_3$ , except that it is arranged according to Lemma 2.9. We then define  $G_5$  in the same way as  $G_4$ , except that we change the way  $M^*$  is determined. Instead of computing  $M^* = \mathbf{Encode}(m_b, u)$ , we sample  $M^* \leftarrow \psi_{\mathcal{M}}$ . In  $G_4$ , however, since  $u$  is sampled from  $\psi_{\mathcal{U}}$  and

| <b>GAMES <math>G_1</math>-<math>G_5</math></b>                   | <b><math>\mathcal{C}^G(r^*, u)</math></b>         |                         |
|--|---|-------------------------|
| 1: $\mathsf{G} \leftarrow (\mathcal{R} \rightarrow \mathcal{U})$ | $\mathcal{C}^G(r^*, u)$                           |                         |
| 2: $r^* \leftarrow \psi_{\mathcal{R}}$                           | $\mathcal{C}^G(r^*, u)$                           |                         |
| 3: $u := \mathsf{G}(r^*)$  | $\mathcal{C}^G(r^*, u)$                           |                         |
| 4: $\mathsf{F} \leftarrow (\mathcal{R} \rightarrow \mathcal{U})$ | $\mathcal{C}^G(r^*, u)$                           |                         |
| 5: $u \leftarrow \psi_{\mathcal{U}}$                             | $\mathcal{C}^G(r^*, u)$                           |                         |
| 6: $\mathsf{G} := \mathsf{F}(r^* := u)$                          | $\mathcal{C}^G(r^*, u)$                           |                         |
| 7: $w \leftarrow \mathcal{C}^G(r^*, u)$                          | $\mathcal{C}^G(r^*, u)$                           |                         |
| 8: $w \leftarrow \mathcal{C}^F(r^*, u)$                          | $\mathcal{C}^G(r^*, u)$                           |                         |
| 9: $T \leftarrow \mathcal{D}^F(r^*, u)$                          | $\mathcal{C}^G(r^*, u)$                           |                         |
| 10: <b>return</b> $w$  | $\mathcal{C}^G(r^*, u)$                           |                         |
| 11: <b>return</b> $\llbracket r^* \in T \rrbracket$              | $\mathcal{C}^G(r^*, u)$                           |                         |
|  | $\mathcal{C}^G(r^*, u)$                           |                         |
|  | 1: $(pk, sk) \leftarrow \mathsf{Gen}(1^\lambda)$  |                         |
|  | 2: $(m_0, m_1) \leftarrow \mathcal{A}_0^G(pk)$    |                         |
|  | 3: $b \leftarrow \{0, 1\}$ ;                      | $\mathcal{C}^G(r^*, u)$ |
|  | 4: $M^* = \mathsf{Encode}(m_b, u)$ ;              | $\mathcal{C}^G(r^*, u)$ |
|  | 5: $M^* \leftarrow \psi_{\mathcal{M}}$ ;          | $\mathcal{C}^G(r^*, u)$ |
|  | 6: $c^* \leftarrow \mathsf{Enc}(pk, M^*; r^*)$    | $\mathcal{C}^G(r^*, u)$ |
|  | 7: $b' \leftarrow \mathcal{A}_1^G(pk, c^*)$       | $\mathcal{C}^G(r^*, u)$ |
|  | 8: <b>return</b> $\llbracket b = b' \rrbracket$   | $\mathcal{C}^G(r^*, u)$ |
|  | $\mathcal{D}^F(r, u)$                             | $\mathcal{C}^G(r^*, u)$ |
|  | 1: $i \leftarrow \{1, \dots, q_G\}$               | $\mathcal{D}^F(r, u)$   |
|  | 2: Run $\mathcal{C}^F(r^*, u)$ till $i$ -th query | $\mathcal{D}^F(r, u)$   |
|  | 3: $T \leftarrow$ measure $\mathsf{F}$ -query     | $\mathcal{D}^F(r, u)$   |
|  | 4: <b>return</b> $T$                              | $\mathcal{D}^F(r, u)$   |

Figure 10: GAMES  $G_1$ - $G_5$  for the proof of Theorem 3.7

used only for  $\mathsf{Encode}(m_b, u)$ , the message-hiding property of SOTP ensures that  $M^* = \mathsf{Encode}(m_b, u)$  is distributed according to  $\psi_{\mathcal{M}}$ . Hence,

$$\Pr[G_4^A \Rightarrow 1] = \Pr[G_5^A \Rightarrow 1].$$

GAME  $G_6$ . We define  $G_6$  in the same way as  $G_5$ , except that the challenger's procedure is rearranged as shown in Figure 11. Since this change is only conceptual, we have

$$\Pr[G_5^A \Rightarrow 1] = \Pr[G_6^E \Rightarrow 1].$$

Let  $\text{BAD}$  be the event that there exists  $r \in T$  with  $r \neq r^*$  and  $M := \mathsf{MRec}(pk, r, c^*) \in \mathcal{M}$ . Then,

$$\begin{aligned} \Pr[G_6^E \Rightarrow 1] &= \Pr[G_6^E \Rightarrow 1 \wedge \text{BAD in } G_6^E] + \Pr[G_6^E \Rightarrow 1 \wedge \neg \text{BAD in } G_6^E] \\ &\leq \Pr[\text{BAD in } G_6^E] + \Pr[G_6^E \Rightarrow 1 \wedge \neg \text{BAD in } G_6^E] \end{aligned}$$

GAME  $G_{7a}$ . We define  $G_{7a}$  by wrapping the generation of  $(M^*, r^*)$  and  $c^*$  into the adversary  $\mathcal{B}$ . Also, unlike  $G_6$ , after  $\mathcal{E}$  outputs  $T$ ,  $\mathcal{B}$  searches for some  $r \in T$  with  $r \neq r^*$  such that  $M = \mathsf{MRec}(pk, r, c^*) \in \mathcal{M}$ , and outputs  $(M, M^*, r, r^*)$  if such an  $r$  exists. Otherwise, it samples random  $(M, r)$  and outputs  $(M, M^*, r, r^*)$ . Suppose such an  $r$  exists. By AC-MR, the pair  $(M, r)$  is a valid pre-image of  $c^*$ . Since  $(M^*, r^*)$  is also a pre-image of  $c^*$  and  $r \neq r^*$ , the tuple  $(M, M^*, r, r^*)$  is a valid solution to the injectivity game. Therefore, we have

$$\Pr[\text{BAD in } G_6^E] = \Pr[G_{7a}^B \Rightarrow 1].$$

Moreover, by definition,  $G_{7a}$  is exactly the  $\text{INJ}$  game for PKE run with adversary  $\mathcal{B}$ ; hence

$$\Pr[G_{7a}^B \Rightarrow 1] = \mathsf{Adv}_{\text{PKE}}^{\text{INJ}}(\mathcal{B}).$$

GAME  $G_{7b}$ . We define  $G_{7b}$  in the same way as  $G_6$ , except that algorithm  $\mathcal{C}$  explicitly outputs  $(M, r)$  and the game returns 1 if  $(M^*, r^*) = (M, r)$ . (see Figure 11). If there exist  $r^* \in T$  and there does not exist

GAMES  $G_6$

- 1:  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$
- 2:  $(M^*, r^*) \leftarrow \psi_{\mathcal{M}} \times \psi_{\mathcal{R}}$
- 3:  $c^* \leftarrow \text{Enc}(pk, M^*; r^*)$
- 4:  $T \leftarrow \mathcal{E}(pk, c^*)$
- 5: **return**  $\llbracket r^* \in T \rrbracket$

$\mathcal{E}(pk, c^*)$

- 1:  $i \leftarrow \{1, \dots, q_{\mathcal{G}}\}$
- 2: Run until  $i$ -th  $\mathsf{F}$ -query:
- 3:  $\mathcal{A}_1^{\mathsf{F}}(pk)$
- 4:  $\mathcal{A}_2^{\mathsf{F}}(pk, c^*)$
- 5:  $T \leftarrow \text{measure } \mathsf{F}\text{-query}$
- 6: **return**  $T$

GAMES  $G_{7a}$

- 1:  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$
- 2:  $(M, M^*, r, r^*) \leftarrow \mathcal{B}(pk)$
- 3:  $c = \text{Enc}(pk, M; r)$
- 4:  $c' = \text{Enc}(pk, M^*; r^*)$
- 5: **return**  $\llbracket (M, M^*, r, r^*) \in \mathcal{M}^2 \times \mathcal{R}^2 \wedge (M, r) \neq (M^*, r^*) \wedge c = c' \rrbracket$

$\mathcal{B}(pk)$

- 1:  $(M^*, r^*) \leftarrow \psi_{\mathcal{M}} \times \psi_{\mathcal{R}}$
- 2:  $c^* \leftarrow \text{Enc}(pk, M^*; r^*)$
- 3:  $T \leftarrow \mathcal{E}(pk, c^*)$
- 4: **for**  $r \in T$  with  $r \neq r^*$  **do**
- 5:   **if**  $M = \text{MRec}(pk, r, c^*) \in \mathcal{M}$
- 6:     **return**  $(M, M^*, r, r^*)$
- 7:  $(M, r) \leftarrow \psi_{\mathcal{M}} \times \psi_{\mathcal{R}}$
- 8: **return**  $(M, r, M^*, r^*)$

GAMES  $G_{7b}$

- 1:  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$
- 2:  $(M^*, r^*) \leftarrow \psi_{\mathcal{M}} \times \psi_{\mathcal{R}}$
- 3:  $c^* \leftarrow \text{Enc}(pk, M^*; r^*)$
- 4:  $M \leftarrow \mathcal{C}(pk, c^*)$
- 5: **return**  $\llbracket M^* = M \rrbracket$

$\mathcal{C}(pk, c^*)$

- 1:  $T \leftarrow \mathcal{E}(pk, c^*)$
- 2: **for**  $r \in T$  **do**
- 3:   **if**  $M = \text{MRec}(pk, r, c^*) \in \mathcal{M}$
- 4:     **return**  $(M, r)$
- 5: **return**  $(M, r) \leftarrow \psi_{\mathcal{M}} \times \psi_{\mathcal{R}}$

Figure 11: GAMES  $G_6$ - $G_7$  for the proof of Theorem 3.7

$r \in T$  with  $r \neq r^*$  and  $M = \text{MRec}(pk, r, c^*) \in \mathcal{M}$ , by VC-MR, we can recover  $r^* = \text{MRec}(pk, r^*, c^*)$ . Therefore, we have

$$\Pr[G_6^{\mathcal{E}} \Rightarrow 1 \wedge \neg \text{BAD in } G_6^{\mathcal{E}}] = \Pr[G_{7b}^{\mathcal{C}} \Rightarrow 1].$$

By definition,  $G_{7b}$  is exactly the OW-CPA game for PKE run with adversary  $\mathcal{C}$ ; hence

$$\Pr[G_7^{\mathcal{C}} \Rightarrow 1] = \text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{C}).$$

Combining all (in)equalities and bounds, we obtain

$$\text{Adv}_{\text{ACWC}_2[\text{PKE}, \text{SOTP}, \mathcal{G}]}^{\text{IND-CPA}}(\mathcal{A}) \leq 2q_{\mathcal{G}} \sqrt{\text{Adv}_{\text{PKE}}^{\text{INJ}}(\mathcal{B}) + \text{Adv}_{\text{PKE}}^{\text{OW-CPA}}(\mathcal{C})},$$

which concludes the proof.  $\square$

### 3.2.3 Spreadness of $\text{PKE}'$

**Theorem 3.8.** If  $\text{PKE}$  is (weakly)  $\gamma$ -spread, SOTP has the message hiding property, and  $\mathsf{G}$  is modeled as a random oracle, then  $\text{PKE}' = \text{ACWC}_2[\text{PKE}, \text{SOTP}, \mathsf{G}]$  is (weakly)  $\gamma$ -spread.

*Proof.* For a fixed  $(pk, sk)$  and  $m$ , we consider the probability  $\Pr_{R \leftarrow \mathcal{R}', \mathsf{G}}[c = \text{Enc}'(pk, m; R)]$  for any ciphertext  $c$ . Since  $\mathsf{G}$  is modeled as a random oracle, the probability is taken over the random choice of  $\mathsf{G}$ . Given that  $r$  is sampled as  $r \leftarrow \psi_{\mathcal{R}}$  using the randomness  $R \leftarrow \mathcal{R}'$ , the probability can be rewritten as

$$\Pr_{R \leftarrow \mathcal{R}', \mathsf{G}}[c = \text{Enc}'(pk, m; R)] = \Pr_{r \leftarrow \psi_{\mathcal{R}}, \mathsf{G}}[c = \text{Enc}(pk, \text{Encode}(m, \mathsf{G}(r)); r)].$$

By the law of total probability on possible  $r \leftarrow \psi_{\mathcal{R}}$ , we have:

$$\Pr_{r \leftarrow \psi_{\mathcal{R}}, \mathsf{G}}[c = \text{Enc}(pk, \text{Encode}(m, \mathsf{G}(r)); r)] = \sum_{r_i \in \mathcal{R}} \Pr_{\mathsf{G}}[c = \text{Enc}(pk, \text{Encode}(m, \mathsf{G}(r_i)); r_i)] \Pr_{r \leftarrow \psi_{\mathcal{R}}}[r = r_i].$$

Since  $\mathsf{G}(r_i)$  is  $\psi_{\mathcal{U}}$ -distributed, the message hiding property of SOTP ensures that the output  $M = \text{Encode}(m, \mathsf{G}(r_i))$  is  $\psi_{\mathcal{M}}$ -distributed over the random choice of  $\mathsf{G}$ :

$$\begin{aligned} & \sum_{r_i \in \mathcal{R}} \Pr_{\mathsf{G}}[c = \text{Enc}(pk, \text{Encode}(m, \mathsf{G}(r_i)); r_i)] \Pr_{r \leftarrow \psi_{\mathcal{R}}}[r = r_i] \\ &= \sum_{r_i \in \mathcal{R}} \Pr_{u \leftarrow \psi_{\mathcal{U}}}[c = \text{Enc}(pk, \text{Encode}(m, u); r_i)] \Pr_{r \leftarrow \psi_{\mathcal{R}}}[r = r_i] \\ &= \sum_{r_i \in \mathcal{R}} \Pr_{M \leftarrow \psi_{\mathcal{M}}}[c = \text{Enc}(pk, M; r_i)] \Pr_{r \leftarrow \psi_{\mathcal{R}}}[r = r_i]. \end{aligned}$$

For the ease of analysis, we define an indicator function  $\mathbf{I}(pk, M, r, c) = \llbracket c == \text{Enc}(pk, M; r) \rrbracket$ . Then,

$$\begin{aligned} & \sum_{r_i \in \mathcal{R}} \Pr_{M \leftarrow \psi_{\mathcal{M}}}[c = \text{Enc}(pk, M; r_i)] \Pr_{r \leftarrow \psi_{\mathcal{R}}}[r = r_i] \\ &= \sum_{r_i \in \mathcal{R}} \sum_{M_j \in \mathcal{M}} \mathbf{I}(pk, M_j, r_i, c) \Pr_{M \leftarrow \psi_{\mathcal{M}}}[M = M_j] \Pr_{r \leftarrow \psi_{\mathcal{R}}}[r = r_i] \\ &= \sum_{M_j \in \mathcal{M}} \sum_{r_i \in \mathcal{R}} \mathbf{I}(pk, M_j, r_i, c) \Pr_{r \leftarrow \psi_{\mathcal{R}}}[r = r_i] \Pr_{M \leftarrow \psi_{\mathcal{M}}}[M = M_j] \\ &= \sum_{M_j \in \mathcal{M}} \Pr_{r \leftarrow \psi_{\mathcal{R}}}[c = \text{Enc}(pk, M_j; r)] \Pr_{M \leftarrow \psi_{\mathcal{M}}}[M = M_j]. \end{aligned}$$

Considering  $\Pr_{r \leftarrow \psi_{\mathcal{R}}}[c = \text{Enc}(pk, M_j; r)]$  as the  $\gamma$ -spreadness of  $\text{PKE}$  on any message  $M_j$ , the  $\gamma'$ -spreadness of  $\text{PKE}'$  is upper-bounded as follows:

$$\begin{aligned} \Pr_{R \leftarrow \mathcal{R}', \mathsf{G}}[c = \text{Enc}'(pk, m; R)] &= \sum_{M_j \in \mathcal{M}} \Pr_{r \leftarrow \psi_{\mathcal{R}}}[c = \text{Enc}(pk, M_j; r)] \cdot \Pr_{M \leftarrow \psi_{\mathcal{M}}}[M = M_j] \\ &\leq 2^{-\gamma} \cdot \sum_{M_j \in \mathcal{M}} \Pr_{M \leftarrow \psi_{\mathcal{M}}}[M = M_j] = 2^{-\gamma}. \end{aligned}$$

By averaging over  $(pk, sk)$ , the weak  $\gamma'$ -spreadness of  $\text{PKE}'$  is also obtained.  $\square$

## 4 IND-CCA Secure KEM from ACWC<sub>2</sub>

### 4.1 FO Transform with Re-encryption

One can apply the Fujisaki-Okamoto transformation  $\text{FO}^\perp$  to the IND-CPA secure  $\text{PKE}'$ , as shown in Figure 5, to obtain an IND-CCA secure KEM. Figure 12 shows the resultant KEM :=  $\text{FO}^\perp[\text{PKE}', \text{H}] = (\text{Gen}, \text{Encap}, \text{Decap})$ , where  $\text{H}$  is a hash function (modeled as a random oracle). Regarding the correctness error of KEM, KEM preserves the worst-case correctness error of  $\text{PKE}'$ , as  $\text{Decap}$  works correctly as long as  $\text{Dec}'$  is performed correctly. Regarding the IND-CCA security of KEM, we can use the previous results [20] and [15], which are stated in Theorems 4.1 and 4.2, respectively. By combining these results with Theorems 3.6 and 3.7, we can achieve the IND-CCA security of KEM in the classical/quantum random oracle model. In the case of the quantum random oracle model (QROM), we need to further use the fact that IND-CPA security generically implies OW-CPA security.

| <u>Encap</u> ( $pk$ )  | <u>Decap</u> ( $sk, c$ )  |
|--|---|
| 1: $m \leftarrow \mathcal{M}$                                | 1: $m' := \text{Dec}'(sk, c)$   |
| 2: $(R, K) := \text{H}(m)$                                   | - $M' = \text{Dec}(sk, c)$  |
| 3: $c := \text{Enc}'(pk, m; R)$                              | - $r' = \text{RRec}(pk, M', c)$   |
| - $r \leftarrow \psi_{\mathcal{R}}$ using the randomness $R$ | - $m' = \text{Decode}(M', \text{G}(r'))$  |
| - $M := \text{Encode}(m, \text{G}(r))$                       | - <b>if</b> $r' \notin \mathcal{R}$ or $m' = \perp$ , <b>return</b> $\perp$           |
| - $c := \text{Enc}(pk, M; r)$                                | - <b>return</b> $m'$  |
| 4: <b>return</b> $(K, c)$                                    | 2: $(R', K') := \text{H}(m')$   |
|  | 3: <b>if</b> $m' = \perp$ or $c \neq \text{Enc}'(pk, m'; R')$ , <b>return</b> $\perp$ |
|  | 4: <b>else, return</b> $K'$   |

Figure 12:  $\text{KEM} = \text{FO}^\perp[\text{PKE}', \text{H}]$

**Theorem 4.1** (IND-CPA security of  $\text{PKE}' \xrightarrow{\text{ROM}}$  IND-CCA security of KEM [20]). Let  $\text{PKE}'$  be a public-key encryption scheme with a message space  $\mathcal{M}$ . Let  $\text{PKE}'$  has (worst-case) correctness error  $\delta$  and is (weakly)  $\gamma$ -spread. For any adversary  $\mathcal{A}$  making at most  $q_D$  decapsulation and  $q_H$  hash queries, against the IND-CCA security of KEM, there exists an adversary  $\mathcal{B}$  against the IND-CPA security of  $\text{PKE}'$  with

$$\text{Adv}_{\text{KEM}}^{\text{IND-CCA}}(\mathcal{A}) \leq 2(\text{Adv}_{\text{PKE}'}^{\text{IND-CPA}}(\mathcal{B}) + \frac{q_H}{|\mathcal{M}|}) + q_D 2^{-\gamma} + q_H \delta,$$

where the running time of  $\mathcal{B}$  is about that of  $\mathcal{A}$ .

**Theorem 4.2** (OW-CPA security of  $\text{PKE}' \xrightarrow{\text{QROM}}$  IND-CCA security of KEM [15]). Let  $\text{PKE}'$  have (worst-case) correctness error  $\delta$  and be (weakly)  $\gamma$ -spread. For any quantum adversary  $\mathcal{A}$ , making at most  $q_D$  decapsulation and  $q_H$  (quantum) hash queries against the IND-CCA security of KEM, there exists a quantum adversary  $\mathcal{B}$  against the OW-CPA security of  $\text{PKE}'$  with

$$\text{Adv}_{\text{KEM}}^{\text{IND-CCA}}(\mathcal{A}) \leq 2q \sqrt{\text{Adv}_{\text{PKE}'}^{\text{OW-CPA}}(\mathcal{B})} + 24q^2 \sqrt{\delta} + 24q \sqrt{q q_D} \cdot 2^{-\gamma/4},$$

where  $q := 2(q_H + q_D)$  and  $\text{Time}(\mathcal{B}) \approx \text{Time}(\mathcal{A}) + O(q_H \cdot q_D \cdot \text{Time}(\text{Enc}) + q^2)$ .

## 4.2 FO-Equivalent Transform Without Re-encryption

The aforementioned  $\text{FO}^\perp$  requires the Decap algorithm to perform re-encryption to check if ciphertext  $c$  is well-formed. Using  $m'$  as the result of  $\text{Dec}'(sk, c)$ , a new randomness  $R'$  is obtained from  $\text{H}(m')$ , and  $\text{Enc}'(pk, m'; R')$  is computed and compared with the (decrypted) ciphertext  $c$ . Even if  $m'$  is the same as  $m$  used in Encap, it does not guarantee that  $\text{Enc}'(pk, m'; R') = c$  without computing  $R'$  and performing re-encryption. In other words, there could exist many other ciphertexts  $\{c_i\}$  (including  $c$  as one of them), all of which are decrypted into the same  $m'$  but generated with distinct randomness  $\{R'\}$ . In  $\text{FO}^\perp$  (and other FO transformations), there is still no way to find the same  $c$  (honestly) generated in Encap other than by comparing  $\text{Enc}'(pk, m'; R')$  and  $c$ . In the context of chosen-ciphertext attacks (using the inequality such as  $c \neq \text{Enc}'(pk, m'; R')$ ), it is well known that decapsulation queries using  $\{c_i\}$  can leak information on  $sk$ , particularly in lattice-based encryption schemes.

However, we demonstrate that  $\text{FO}^\perp$  based on  $\text{ACWC}_2$  can eliminate the need for ciphertext comparison  $c = \text{Enc}'(pk, m'; R')$  in Decap, and instead replace it with a simpler and more efficient comparison  $r' = r''$ . To do this, we first change Decap of Figure 12 into that of Figure 13, which are conceptually identical to each other. Rather, the change has the effect of preventing reaction attacks that can occur by returning distinct output errors of Decap. Next, we suggest the new  $\text{FO}^\perp$  conversion based on  $\text{ACWC}_2$ , denoted as  $\overline{\text{FO}}^\perp$ , as shown in Figure 14. In  $\overline{\text{FO}}^\perp$ ,  $r'$  and  $r''$  are values generated during the execution of Decap, where  $r'$  is the output of  $\text{RRec}(pk, M', c)$  and  $r''$  is computed from the randomness  $R'$  of  $\text{H}(m')$ . The only change compared to  $\text{FO}^\perp$  in Figure 13 is the boxed area, while the remaining parts remain the same. By proving that the two conditions  $r' \notin \mathcal{R}$  and  $c = \text{Enc}'(pk, m'; R')$  are equivalent to the equality  $r' = r''$  (where  $r'' \leftarrow \psi_{\mathcal{R}}$  with the randomness  $R'$ ), we can show that both  $\text{FO}^\perp$  and  $\overline{\text{FO}}^\perp$  work identically and thus achieve the same level of IND-CCA security.

### Decap( $sk, c$ )

```

1:  $M' = \text{Dec}(sk, c)$ 
2:  $r' = \text{RRec}(pk, M', c)$ 
3:  $m' = \text{Decode}(M', \text{G}(r'))$ 
4:  $(R', K') := \text{H}(m')$ 
5: if  $m' \neq \perp$  and  $r' \in \mathcal{R}$  and  $c = \text{Enc}'(pk, m'; R')$ 
6:   return  $K'$ 
7: else
8:   return  $\perp$ 

```

### Decap( $sk, c$ )

```

1:  $M' = \text{Dec}(sk, c)$ 
2:  $r' = \text{RRec}(pk, M', c)$ 
3:  $m' = \text{Decode}(M', \text{G}(r'))$ 
4:  $(R', K') := \text{H}(m')$ 
5:  $r'' \leftarrow \psi_{\mathcal{R}}$  with the randomness  $R'$ 
6: if  $m' \neq \perp$  and  $r' = r''$ 
7:   return  $K'$ 
8: else
9:   return  $\perp$ 

```

Figure 13: Modified KEM =  $\text{FO}^\perp[\text{PKE}', \text{H}]$

Figure 14:  $\text{KEM}' = \overline{\text{FO}}^\perp[\text{PKE}', \text{H}]$

### 4.2.1 Security Proof in the ROM

**Theorem 4.3.** Let KEM be a key encapsulation mechanism defined in Figure 12, and let  $\text{KEM}'$  be another mechanism defined in Figure 14, both constructed based on PKE. Assume that PKE is randomness recoverable and randomness unique, has an average-case correctness error  $\delta$ , and ensures that outputs of Dec always belong to  $\mathcal{M}$  and that SOTP is rigid. For any adversary  $\mathcal{A}$  making at most  $q_D$  decapsulation and  $q_H$  hash queries against the IND-CCA security of  $\text{KEM}'$ , there exists an adversary  $\mathcal{B}$  against the IND-CCA security

of KEM with

$$\text{Adv}_{\text{KEM}'}^{\text{IND-CCA}}(\mathcal{A}) \leq \text{Adv}_{\text{KEM}}^{\text{IND-CCA}}(\mathcal{B}) + (q_{\mathsf{H}} + q_{\mathsf{D}})\delta',$$

where  $\delta' = \delta + \|\psi_{\mathcal{R}}\| \cdot (1 + \sqrt{(\ln |\mathcal{M}'| - \ln \|\psi_{\mathcal{R}}\|)/2})$ , and  $\|\psi_{\mathcal{R}}\| := \sqrt{\sum_r \psi_{\mathcal{R}}(r)^2}$ .

*Proof.* The security proof begins by analyzing hybrid games with a fixed key pair  $(pk, sk)$ . A detailed explanation of the security proof is provided below.

GAME  $G_0$ .  $G_0$  is the original IND-CCA game against KEM' with a fixed key pair  $(pk, sk)$ .

GAME  $G_1$ . In contrast to  $G_0$ , the Decap oracle in  $G_1$  is modified. Instead of returning  $K'$  when  $m' \neq \perp$  and  $r' = r''$ ,  $K'$  is now returned if  $m' \neq \perp$ ,  $r' \in \mathcal{R}$ , and  $c = c'$ . Note that  $G_1$  is the original IND-CCA game against KEM with a fixed key pair  $(pk, sk)$ . For ease of analysis, for each  $sk$  and  $m \in \mathcal{M}$ , we define

$$\begin{aligned} \mathcal{R}'_{\text{bad}}(sk, m) := \{ & R \in \mathcal{R}' : \text{Dec}(sk, \text{Enc}(pk, M; r)) \neq M, \\ & \text{where } r = \text{Sample}(\mathcal{R}; R) \text{ and } M = \text{Encode}(m, G(r)) \}. \end{aligned}$$

and  $\mathcal{R}'_{\text{good}}(sk, m) := \mathcal{R}' \setminus \mathcal{R}'_{\text{bad}}(sk, m)$ . Additionally, we define  $\delta(sk, m) := |\mathcal{R}'_{\text{bad}}(sk, m)|/|\mathcal{R}'|$  and  $\delta_{sk} := \max_{m \in \mathcal{M}} \delta(sk, m)$ .

Assuming  $(R, K) = \mathsf{H}(m) \in \mathcal{R}'_{\text{good}}(sk, m) \times \mathcal{K}$  for all  $m$  that are queried to  $\mathsf{H}$ , we now show that the changes in  $G_1$  do not impact adversary  $\mathcal{A}$ , as the conditions actually imply each other.

Assume that  $m' \neq \perp$  and  $r' = r''$  hold for a ciphertext  $c$  in the Decap oracle. Given the rigidity of the SOTP, the condition  $m' = \text{Decode}(M', G(r')) \neq \perp$  implies  $M' = \text{Encode}(m', G(r'))$ , and thus  $M' = \text{Encode}(m', G(r''))$ . Moreover, since  $r' = r''$  and  $r''$  is sampled from  $\psi_{\mathcal{R}}$  using the randomness  $R'$ , it follows that  $r' \in \mathcal{R}$ . Additionally, since  $M' = \text{Dec}(sk, c)$  is within  $\mathcal{M}$  and  $r' = \text{RRec}(pk, M', c)$  is also in  $\mathcal{R}$ , the RR property of the PKE ensures that  $c = \text{Enc}(pk, M'; r') = \text{Enc}(pk, \text{Encode}(m', G(r'')); r'') = c'$ .

Conversely, assume that  $m' \neq \perp$ ,  $r' \in \mathcal{R}$ , and  $c = c'$  hold in the Decap oracle. Since  $M' = \text{Dec}(sk, c)$  is within  $\mathcal{M}$  and  $r' = \text{RRec}(pk, M', c)$  is in  $\mathcal{R}$ , the RR property of the PKE ensures that  $c = \text{Enc}(pk, M'; r')$ . Additionally, since  $c = c' = \text{Enc}(pk, \text{Encode}(m', G(r'')); r'')$  where  $r''$  is sampled using  $R'$  from the pair  $(R', K') = \mathsf{H}(m')$ , by the definition of  $\mathsf{H}$ , it follows that  $M' = \text{Dec}(sk, c) = \text{Encode}(m', G(r''))$ . Consequently, this implies that  $c = \text{Enc}(pk, M'; r'') = \text{Enc}(pk, M'; r'')$ . Since  $c = \text{Enc}(pk, M'; r') = \text{Enc}(pk, M'; r'')$ , the randomness uniqueness of PKE implies  $r' = r''$ .

| GAMES $G_0$ - $G_1$   | Decap( $sk, c$ )   |
|---|--|
| 1: $\mathsf{G} \leftarrow (\mathcal{R} \rightarrow \mathcal{U})$                      | 1: $M' = \text{Dec}(sk, c)$  |
| 2: $\mathsf{H} \leftarrow (\mathcal{M}' \rightarrow \mathcal{R}' \times \mathcal{K})$ | 2: $r' = \text{RRec}(pk, M', c)$   |
| 3: $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$  | 3: $m' = \text{Decode}(M', G(r'))$   |
| 4: $(K_0, c^*) \leftarrow \text{Encap}(pk)$   | 4: $(R', K') := \mathsf{H}(m')$  |
| 5: $K_1 \leftarrow \mathcal{K}$   | 5: $r'' \leftarrow \psi_{\mathcal{R}}$ with the randomness $R'$  |
| 6: $b \leftarrow \{0, 1\}$  | 6: $c' = \text{Enc}(pk, \text{Encode}(m', G(r'')); r'')$   |
| 7: $b' \leftarrow \mathcal{A}^{\mathsf{G}, \mathsf{H}, \text{Decap}}(pk, c^*, K_b)$   | 7: <b>if</b> $m' \neq \perp$ and $r' = r''$ <b>then</b> $K'$ <b>else</b> $K$ $\text{else}$ $\perp$ $\text{else}$ $\perp$ |
| 8: <b>return</b> $\llbracket b = b' \rrbracket$                                       | 8: <b>if</b> $m' \neq \perp$ and $r' \in \mathcal{R}$ and $c = c'$ <b>then</b> $K'$ <b>else</b> $K$ <b>else</b> $\perp$  |

Figure 15: GAMES  $G_0$ - $G_1$  for the proof of Theorem 4.3

Since these two conditions imply each other, assuming that  $(R, K) = \mathsf{H}(m)$  belongs to  $\mathcal{R}'_{\text{good}}(sk, m) \times \mathcal{K}$  for all  $m$  that are queried to  $\mathsf{H}$ , by Lemma 2.8, the following holds:

$$|\Pr[G_0^{\mathcal{A}} \Rightarrow 1] - \Pr[G_1^{\mathcal{A}} \Rightarrow 1]| \leq (q_{\mathsf{H}} + q_{\mathsf{D}})\delta_{sk} \quad (7)$$

Therefore, by the triangular inequality, the following holds:

$$\begin{aligned} \text{Adv}_{\text{KEM}', sk}^{\text{IND-CCA}}(\mathcal{A}) &= |\Pr[G_0^{\mathcal{A}} \Rightarrow 1] - 1/2| \\ &\leq |\Pr[G_0^{\mathcal{A}} \Rightarrow 1] - \Pr[G_1^{\mathcal{A}} \Rightarrow 1]| + |\Pr[G_1^{\mathcal{A}} \Rightarrow 1] - 1/2| \\ &\leq \text{Adv}_{\text{KEM}, sk}^{\text{IND-CCA}}(\mathcal{A}) + (q_{\mathsf{H}} + q_{\mathsf{D}})\delta_{sk}. \end{aligned}$$

By taking the expectation over  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$  and applying Corollary 3.3, we obtain the desired bound.  $\square$

#### 4.2.2 Security Proof in the QROM

**Theorem 4.4.** Let  $\text{KEM}$  be the key encapsulation mechanism defined in Figure 12, and let  $\text{KEM}'$  be the mechanism defined in Figure 14, both constructed from the same public-key encryption scheme  $\text{PKE}$ . Assume that  $\text{PKE}$  is randomness recoverable and randomness unique, has average-case correctness error  $\delta$ , and ensures that the outputs of  $\text{Dec}$  always lie in  $\mathcal{M}$  and that  $\text{SOTP}$  is rigid. For any quantum adversary  $\mathcal{A}$  making at most  $q_{\mathsf{D}}$  decapsulation queries and  $q_{\mathsf{H}}$  (quantum) hash queries against the IND-CCA security of  $\text{KEM}'$ , there exists a quantum adversary  $\mathcal{B}$  against the IND-CCA security of  $\text{KEM}$  with

$$\text{Adv}_{\text{KEM}'}^{\text{IND-CCA}}(\mathcal{A}) \leq \text{Adv}_{\text{KEM}}^{\text{IND-CCA}}(\mathcal{B}) + 4 \cdot (q_{\mathsf{H}} + q_{\mathsf{D}})\sqrt{\delta'},$$

where  $\delta' = \delta + \|\psi_{\mathcal{R}}\| \cdot (1 + \sqrt{(\ln |\mathcal{M}'| - \ln \|\psi_{\mathcal{R}}\|)/2})$ , and  $\|\psi_{\mathcal{R}}\| := \sqrt{\sum_r \psi_{\mathcal{R}}(r)^2}$ .

*Proof.* The security proof proceeds via a sequence of hybrid games, analyzed for a fixed key pair  $(pk, sk)$ . A detailed explanation of the security proof is provided below.

GAME  $G_0$ .  $G_0$  is the original IND-CCA game against  $\text{KEM}'$  with a fixed key pair  $(pk, sk)$ .

GAME  $G_1$ . Unlike  $G_0$ ,  $G_1$  uses the function  $\mathsf{H}'$  instead of  $\mathsf{H}$ . The function  $\mathsf{H}'$  takes a message  $m$  as input and selects randomness from the set  $\mathcal{R}'_{\text{good}}$ , which consists of all randomness that do not cause decryption errors when encrypting  $m$  under the public key  $pk$ . Specifically, for a fixed secret key  $sk$  and  $m \in \mathcal{M}$ , we define

$$\begin{aligned} \mathcal{R}'_{\text{bad}}(sk, m) &:= \left\{ R \in \mathcal{R}' : \text{Dec}(sk, \text{Enc}(pk, M; r)) \neq M, \right. \\ &\quad \left. \text{where } r = \text{Sample}(\mathcal{R}; R) \text{ and } M = \text{Encode}(m, \mathsf{G}(r)) \right\}. \end{aligned}$$

and  $\mathcal{R}'_{\text{good}}(sk, m) := \mathcal{R}' \setminus \mathcal{R}'_{\text{bad}}(sk, m)$ . The function  $\mathsf{H}'$  is defined as a random function such that  $\mathsf{H}'(m)$  is sampled uniformly from  $\mathcal{R}'_{\text{good}}(sk, m) \times \mathcal{K}$ . We denote by  $\Omega_{\mathsf{H}'}$  the set of all possible choices of  $\mathsf{H}'$ . Finally, we define  $\delta(sk, m) := |\mathcal{R}'_{\text{bad}}(sk, m)|/|\mathcal{R}'|$  and  $\delta_{sk} := \max_{m \in \mathcal{M}} \delta(sk, m)$ .

Note that distinguishing between  $G_0$  and  $G_1$  is equivalent to distinguishing between  $\mathsf{H}$  from  $\mathsf{H}'$ . In particular, we construct an adversary  $\mathcal{B}$  that distinguishes  $\mathsf{H}$  from  $\mathsf{H}'$ . This adversary uses the accessible oracle  $\tilde{\mathsf{H}}$  (either  $\mathsf{H}$  or  $\mathsf{H}'$ ), simulates the view of  $\mathcal{A}$ , and outputs the same results as in games  $G_0$  and  $G_1$ . When  $\tilde{\mathsf{H}} = \mathsf{H}$ ,  $\mathcal{B}^{\tilde{\mathsf{H}}}(sk)$  perfectly simulates  $G_0$ , so  $\Pr[1 \leftarrow \mathcal{B}^{\mathsf{H}}(sk)] = \Pr[G_0^{\mathcal{A}} \Rightarrow 1]$ . Similarly, when  $\tilde{\mathsf{H}} = \mathsf{H}'$ ,  $\mathcal{B}^{\tilde{\mathsf{H}}}(sk)$  simulates  $G_1$ , yielding  $\Pr[1 \leftarrow \mathcal{B}^{\mathsf{H}'}(sk)] = \Pr[G_1^{\mathcal{A}} \Rightarrow 1]$ . Therefore,

$$|\Pr[G_0^{\mathcal{A}} \Rightarrow 1] - \Pr[G_1^{\mathcal{A}} \Rightarrow 1]| = |\Pr[1 \leftarrow \mathcal{B}^{\mathsf{H}}(sk)] - \Pr[1 \leftarrow \mathcal{B}^{\mathsf{H}'}(sk)]|. \quad (8)$$

| <b>GAMES <math>G_0</math>-<math>G_3</math></b>  |                  | <b>Decap(<math>sk, c</math>)</b>  |
|---|------------------|---|
| 1: $\mathbf{G} \leftarrow (\mathcal{R} \rightarrow \mathcal{U})$                      |                  | 1: $M' = \text{Dec}(sk, c)$   |
| 2: $\mathbf{H} \leftarrow (\mathcal{M}' \rightarrow \mathcal{R}' \times \mathcal{K})$ | // $G_0, G_3$    | 2: $r' = \text{RRec}(pk, M', c)$  |
| 3: $\mathbf{H}' \leftarrow \Omega_{\mathbf{H}'}$                                      | // $G_1$ - $G_2$ | 3: $m' = \text{Decode}(M', \mathbf{G}(r'))$   |
| 4: $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$  |                  | 4: $(R', K') := \mathbf{H}(m')$   |
| 5: $(K_0, c^*) \leftarrow \text{Encap}(pk)$   |                  | 5: $r'' \leftarrow \psi_{\mathcal{R}}$ with the randomness $R'$                     |
| 6: $K_1 \leftarrow \mathcal{K}$   |                  | 6: $c' = \text{Enc}(pk, \text{Encode}(m', \mathbf{G}(r'')); r'')$ // $G_2$ - $G_3$  |
| 7: $b \leftarrow \{0, 1\}$  |                  | 7: <b>if</b> $m' \neq \perp$ and $r' = r''$ // $G_0$ - $G_1$                        |
| 8: $b' \leftarrow \mathcal{A}^{\mathbf{G}, \mathbf{H}, \text{Decap}}(pk, c^*, K_b)$   | // $G_0, G_3$    | 8: <b>if</b> $m' \neq \perp$ and $r' \in \mathcal{R}$ and $c = c'$ // $G_2$ - $G_3$ |
| 9: $b' \leftarrow \mathcal{A}^{\mathbf{G}, \mathbf{H}', \text{Decap}}(pk, c^*, K_b)$  | // $G_1$ - $G_2$ | 9: <b>return</b> $K'$   |
| 10: <b>return</b> $\llbracket b = b' \rrbracket$                                      |                  | 10: <b>else, return</b> $\perp$   |

Figure 16: GAMES  $G_0$ - $G_3$  for the proof of Theorem 4.4

| $\mathcal{C}^{\mathbf{N}}(sk)$                              | $\tilde{\mathbf{H}}(m)$  |
|---|--|
| 1: Select $2q_{\mathbf{H}}$ -wise functions $f_1$ and $f_2$ | 1: <b>if</b> $\mathbf{N}(m) = 0$   |
| 2: $b \leftarrow \mathcal{B}^{\mathbf{H}}(sk)$              | 2: $R \leftarrow \mathcal{R}'_{\text{good}}(sk, m)$ with the randomness $f_1(m)$ |
| 3: <b>return</b> $b$  | 3: <b>else</b>   |
|   | 4: $R \leftarrow \mathcal{R}'_{\text{bad}}(sk, m)$ with the randomness $f_1(m)$  |
|   | 5: $K \leftarrow \mathcal{K}$ with randomness $f_2(m)$                           |
|   | 6: <b>return</b> $(R, K)$  |

Figure 17:  $\mathcal{C}^{\mathbf{N}}(sk)$  for the proof of Theorem 4.4

Next, we demonstrate that any adversary  $\mathcal{B}$  distinguishing  $\mathbf{H}$  from  $\mathbf{H}'$  can be converted into an adversary  $\mathcal{C}$  distinguishing  $\mathbf{N}_1$  from  $\mathbf{N}_2$ . Specifically,  $\mathbf{N}_1$  is a function where  $\mathbf{N}_1(m)$  is sampled from the Bernoulli distribution  $B_{\delta_{(sk, m)}}$ , meaning  $\Pr[\mathbf{N}_1(m) = 1] = \delta_{(sk, m)}$  and  $\Pr[\mathbf{N}_1(m) = 0] = 1 - \delta_{(sk, m)}$ . In contrast,  $\mathbf{N}_2$  is a constant function that always outputs 0. For any adversary  $\mathcal{B}^{\mathbf{H}}(sk)$ , we construct an adversary  $\mathcal{C}^{\mathbf{N}}(sk)$  as described in Figure 17. Importantly,  $\mathcal{C}$  simulates  $\tilde{\mathbf{H}} = \mathbf{H}$  when  $\mathbf{N} = \mathbf{N}_1$  and  $\tilde{\mathbf{H}} = \mathbf{H}'$  when  $\mathbf{N} = \mathbf{N}_2$ . Thus,  $\Pr[1 \leftarrow \mathcal{C}^{\mathbf{N}_1}] = \Pr[1 \leftarrow \mathcal{B}^{\mathbf{H}}]$  and  $\Pr[1 \leftarrow \mathcal{C}^{\mathbf{N}_2}] = \Pr[1 \leftarrow \mathcal{B}^{\mathbf{H}'}]$ . Therefore, by Lemma 2.10

$$\left| \Pr[1 \leftarrow \mathcal{B}^{\mathbf{H}}(sk)] - \Pr[1 \leftarrow \mathcal{B}^{\mathbf{H}'}(sk)] \right| \quad (9)$$

$$= \left| \Pr[1 \leftarrow \mathcal{C}^{\mathbf{N}_1}(sk)] - \Pr[1 \leftarrow \mathcal{C}^{\mathbf{N}_2}(sk)] \right| \leq 2 \cdot (q_{\mathbf{H}} + q_{\mathbf{D}}) \sqrt{\delta_{sk}}. \quad (10)$$

Therefore, by combining Equations (8)–(10),

$$\left| \Pr[G_0^{\mathcal{A}} \Rightarrow 1] - \Pr[G_1^{\mathcal{A}} \Rightarrow 1] \right| \leq 2 \cdot (q_{\mathbf{H}} + q_{\mathbf{D}}) \sqrt{\delta_{sk}}. \quad (11)$$

GAME  $G_2$ . In contrast to  $G_1$ , the Decap oracle in  $G_2$  is modified. Instead of returning  $K'$  when  $m' \neq \perp$  and  $r' = r''$ ,  $K'$  is now returned if  $m' \neq \perp$ ,  $r' \in \mathcal{R}$ , and  $c = c'$ . We can show that this modification does not affect the adversary  $\mathcal{A}$  by proving that these two conditions actually imply each other.

Assume that  $m' \neq \perp$  and  $r' = r''$  hold for a ciphertext  $c$  in the Decap oracle. Given the rigidity of the SOTP, the condition  $m' = \text{Decode}(M', G(r')) \neq \perp$  implies  $M' = \text{Encode}(m', G(r'))$ , and thus  $M' = \text{Encode}(m', G(r''))$ . Moreover, since  $r' = r''$  holds and  $r''$  is sampled from  $\psi_{\mathcal{R}}$  using the randomness  $R'$ , it follows that  $r' \in \mathcal{R}$ . Additionally, since  $M' = \text{Dec}(sk, c)$  is within  $\mathcal{M}$  and  $r' = \text{RRec}(pk, M', c)$  is also in  $\mathcal{R}$ , the RR property of the PKE ensures that  $c = \text{Enc}(pk, M'; r') = \text{Enc}(pk, \text{Encode}(m', G(r'')); r'') = c'$ .

Conversely, assume that  $m' \neq \perp$ ,  $r' \in \mathcal{R}$ , and  $c = c'$  hold in the Decap oracle. Since  $M' = \text{Dec}(sk, c)$  is within  $\mathcal{M}$  and  $r' = \text{RRec}(pk, M', c)$  is in  $\mathcal{R}$ , the RR property of the PKE ensures that  $c = \text{Enc}(pk, M'; r')$ . Additionally, since  $c = c' = \text{Enc}(pk, \text{Encode}(m', G(r'')); r'')$  where  $r''$  is sampled using  $R'$  from the pair  $(R', K') = \mathsf{H}(m')$ , by the definition of  $\mathsf{H}$ , it follows that  $M' = \text{Dec}(sk, c) = \text{Encode}(m', G(r''))$ . Consequently, this implies that  $c = \text{Enc}(pk, M'; r'')$ . Since  $c = \text{Enc}(pk, M'; r') = \text{Enc}(pk, M'; r'')$ , the randomness uniqueness of PKE implies  $r' = r''$ .

Since these two conditions imply each other, the following holds:

$$|\Pr[G_1^{\mathcal{A}} \Rightarrow 1] - \Pr[G_2^{\mathcal{A}} \Rightarrow 1]| = 0. \quad (12)$$

GAME  $G_3$ . Unlike  $G_2$ ,  $G_3$  uses the function  $\mathsf{H}$  instead of  $\mathsf{H}'$ . Note that  $G_3$  is the original IND-CCA game against KEM with a fixed key pair  $(pk, sk)$ . By the similar analysis between  $G_0$  and  $G_1$ , the following holds:

$$|\Pr[G_2^{\mathcal{A}} \Rightarrow 1] - \Pr[G_3^{\mathcal{A}} \Rightarrow 1]| \leq 2 \cdot (q_{\mathsf{H}} + q_{\mathsf{D}}) \sqrt{\delta_{sk}}. \quad (13)$$

By combining Equations (11)-(13) with the triangle inequality, the following holds:

$$\begin{aligned} \text{Adv}_{\text{KEM}', sk}^{\text{IND-CCA}}(\mathcal{A}) &= |\Pr[G_0^{\mathcal{A}} \Rightarrow 1] - 1/2| \\ &\leq |\Pr[G_0^{\mathcal{A}} \Rightarrow 1] - \Pr[G_1^{\mathcal{A}} \Rightarrow 1]| + |\Pr[G_1^{\mathcal{A}} \Rightarrow 1] - \Pr[G_2^{\mathcal{A}} \Rightarrow 1]| \\ &\quad + |\Pr[G_2^{\mathcal{A}} \Rightarrow 1] - \Pr[G_3^{\mathcal{A}} \Rightarrow 1]| + |\Pr[G_3^{\mathcal{A}} \Rightarrow 1] - 1/2| \\ &\leq \text{Adv}_{\text{KEM}, sk}^{\text{IND-CCA}}(\mathcal{A}) + 4 \cdot (q_{\mathsf{H}} + q_{\mathsf{D}}) \sqrt{\delta_{sk}}. \end{aligned}$$

By taking the expectation over  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$  and applying corollary 3.3 yields the required bound of the theorem.  $\square$

## 5 NTRU+

### 5.1 GenNTRU $[\psi_1^n]$ (=PKE)

Figure 18 defines GenNTRU $[\psi_1^n]$  relative to the distribution  $\psi_1^n$  over  $R_q$ . Since GenNTRU $[\psi_1^n]$  must satisfy both MR and RR for our ACWC<sub>2</sub> transformation, Figure 18 also includes two auxiliary algorithms, RRec and MRec. We observe that RRec( $\mathbf{h}, \mathbf{m}, \mathbf{c}$ ) is required during ACWC<sub>2</sub>, because  $\mathbf{r}$  must be recovered from a ciphertext  $\mathbf{c}$  once the corresponding message  $\mathbf{m}$  is obtained. The RR property ensures that this randomness-recovery process works well, because for a ciphertext  $\mathbf{c} = \text{Enc}(\mathbf{h}, \mathbf{m}, \mathbf{r}) = \mathbf{h}\mathbf{r} + \mathbf{m}$ , we have  $\text{RRec}(\mathbf{h}, \mathbf{m}, \mathbf{c}) = (\mathbf{c} - \mathbf{m})\mathbf{h}^{-1} = \mathbf{r} \in \mathcal{R}$ . On the other hand, MRec( $\mathbf{h}, \mathbf{r}, \mathbf{c}$ ) is used only in the IND-CPA security proof of the ACWC<sub>2</sub>-transformed scheme. The security analysis requires that for a challenge ciphertext  $\mathbf{c}^* = \text{Enc}(\mathbf{h}, \mathbf{m}^*, \mathbf{r}^*) = \mathbf{h}\mathbf{r}^* + \mathbf{m}^*$ , the algorithm MRec( $\mathbf{h}, \mathbf{r}^*, \mathbf{c}^*$ ) returns the corresponding message  $\mathbf{m}^*$  if the queried randomness  $\mathbf{r}^*$  was used for  $\mathbf{c}^*$ . The MR property guarantees that once  $\mathbf{r}^*$  is given, MRec( $\mathbf{h}, \mathbf{r}^*, \mathbf{c}^*$ ) =  $\mathbf{c}^* - \mathbf{h}\mathbf{r}^* = \mathbf{m}^* \in \mathcal{M}$ .

|  |   |
|--|---|
| <u>Gen</u> ( $1^\lambda$ )                             | <u>Enc</u> ( $\mathbf{h}, \mathbf{m} \leftarrow \psi_1^n; \mathbf{r} \leftarrow \psi_1^n$ ) |
| 1: <b>repeat</b>                                       |   |
| 2: $\mathbf{f}' \leftarrow \psi_1^n$                   | 1: <b>return</b> $\mathbf{c} = \mathbf{h}\mathbf{r} + \mathbf{m}$                           |
| 3: $\mathbf{f} = 3\mathbf{f}' + 1$                     | <u>Dec</u> ( $\mathbf{f}, \mathbf{c}$ )   |
| 4: <b>until</b> $\mathbf{f}$ is invertible in $R_q$    | 1: <b>return</b> $\mathbf{m} = (\mathbf{c}\mathbf{f} \bmod q) \bmod 3$                      |
| 5: <b>repeat</b>                                       | <u>RRec</u> ( $\mathbf{h}, \mathbf{m}, \mathbf{c}$ )  |
| 6: $\mathbf{g} \leftarrow \psi_1^n$                    | 1: <b>return</b> $\mathbf{r} = (\mathbf{c} - \mathbf{m})\mathbf{h}^{-1}$                    |
| 7: <b>until</b> $\mathbf{g}$ is invertible in $R_q$    | <u>MRec</u> ( $\mathbf{h}, \mathbf{r}, \mathbf{c}$ )  |
| 8: $\mathbf{h} = 3\mathbf{g}\mathbf{f}^{-1}$           | 1: <b>return</b> $\mathbf{m} = \mathbf{c} - \mathbf{h}\mathbf{r}$                           |
| 9: <b>return</b> $(pk, sk) = (\mathbf{h}, \mathbf{f})$ |   |

Figure 18: GenNTRU $[\psi_1^n]$  with average-case correctness error

#### 5.1.1 Security Proofs

**Theorem 5.1** (OW-CPA security of GenNTRU $[\psi_1^n]$ ). For any adversary  $\mathcal{A}$ , there exist adversaries  $\mathcal{B}$  and  $\mathcal{C}$  such that

$$\text{Adv}_{\text{GenNTRU}[\psi_1^n]}^{\text{OW-CPA}}(\mathcal{A}) \leq \text{Adv}_{n,q,\psi_1^n}^{\text{NTRU}}(\mathcal{B}) + \text{Adv}_{n,q,\psi_1^n}^{\text{RLWE}}(\mathcal{C}).$$

*Proof.* We complete our proof through a sequence of games  $G_0$  to  $G_1$ . Let  $\mathcal{A}$  be the adversary against the OW-CPA security experiment.

GAME  $G_0$ . In  $G_0$ , we have the original OW-CPA game with GenNTRU $[\psi_1^n]$ . By the definition of the advantage function of the adversary  $\mathcal{A}$  against the OW-CPA game, we have

$$\text{Adv}_{\text{GenNTRU}[\psi_1^n]}^{\text{OW-CPA}}(\mathcal{A}) = \Pr[G_0^{\mathcal{A}} \Rightarrow 1].$$

GAME  $G_1$ . In  $G_1$ , the public key  $\mathbf{h}$  in Gen is replaced by  $\mathbf{h} \leftarrow R_q$ . Therefore, distinguishing  $G_1$  from  $G_0$  is equivalent to solving the NTRU $_{n,q,\psi_1^n}$  problem. More precisely, there exists an adversary  $\mathcal{B}$  with the same running time as  $\mathcal{A}$  such that

$$|\Pr[G_0^{\mathcal{A}} \Rightarrow 1] - \Pr[G_1^{\mathcal{A}} \Rightarrow 1]| \leq \text{Adv}_{n,q,\psi_1^n}^{\text{NTRU}}(\mathcal{B}).$$

Since  $\mathbf{h}$  is now uniformly random in  $R_q$ ,  $G_1$  is equivalent to solving an  $\text{RLWE}_{n,q,\psi_1^n}$  problem. Therefore,

$$\Pr[G_1^A \Rightarrow 1] = \text{Adv}_{n,q,\psi_1^n}^{\text{RLWE}}(\mathcal{C}).$$

Combining all the probabilities completes the proof.  $\square$

### 5.1.2 Average-Case Correctness Error

We analyze the average-case correctness error  $\delta$  relative to the distribution  $\psi_{\mathcal{M}} = \psi_{\mathcal{R}} = \psi_1^n$  using the template provided in [31]. We can expand  $\mathbf{cf}$  in the decryption algorithm as follows:

$$\mathbf{cf} = (\mathbf{hr} + \mathbf{m})\mathbf{f} = (3\mathbf{gf}^{-1}\mathbf{r} + \mathbf{m})(3\mathbf{f}' + 1) = 3(\mathbf{gr} + \mathbf{mf}') + \mathbf{m}.$$

For a polynomial  $\mathbf{p}$  in  $R_q$ , let  $\mathbf{p}_i$  be the  $i$ -th coefficient of  $\mathbf{p}$ , and let  $|\mathbf{p}_i|$  denote the absolute value of  $\mathbf{p}_i$ . Then,  $((\mathbf{cf})_i \bmod q) \bmod 3 = \mathbf{m}_i$  if the following inequality holds:

$$|3(\mathbf{gr} + \mathbf{mf}') + \mathbf{m}|_i \leq \frac{q-1}{2},$$

where all coefficients of each polynomial are distributed according to  $\psi_1^n$ . Let  $\epsilon_i$  be

$$\epsilon_i = \Pr \left[ |3(\mathbf{gr} + \mathbf{mf}') + \mathbf{m}|_i \leq \frac{q-1}{2} \right].$$

Assuming that each coefficient is independent, we have

$$\Pr [\text{Dec}(sk, \text{Enc}(pk, m)) \neq m] = 1 - \prod_{i=0}^{n-1} \epsilon_i. \quad (14)$$

Because the coefficients of  $\mathbf{m}$  have size at most one,

$$\begin{aligned} \epsilon_i &= \Pr \left[ |3(\mathbf{gr} + \mathbf{mf}') + \mathbf{m}|_i \leq \frac{q-1}{2} \right] \\ &\geq \Pr \left[ |3(\mathbf{gr} + \mathbf{mf}')|_i + |\mathbf{m}|_i \leq \frac{q-1}{2} \right] \\ &\geq \Pr \left[ |3(\mathbf{gr} + \mathbf{mf}')|_i + 1 \leq \frac{q-1}{2} \right] \\ &= \Pr \left[ |\mathbf{gr} + \mathbf{mf}'|_i \leq \frac{q-3}{6} \right] := \epsilon'_i. \end{aligned}$$

Therefore,

$$\Pr [\text{Dec}(sk, \text{Enc}(pk, m)) \neq m] = 1 - \prod_{i=0}^n \epsilon_i \leq 1 - \prod_{i=0}^n \epsilon'_i := \delta.$$

Now, we analyze  $\epsilon'_i = \Pr \left[ |\mathbf{gr} + \mathbf{mf}'|_i \leq \frac{q-3}{6} \right]$ . To achieve this, we analyze the distribution of  $\mathbf{gr} + \mathbf{mf}'$ . Following the analysis in [31], we observe that for  $i \in [n/2, n]$ , the degree- $i$  coefficient of  $\mathbf{gr} + \mathbf{mf}'$  is the sum of  $n$  independent random variables:

$$c = ba + b'(a + a') \in \{0, \pm 1, \pm 2, \pm 3\}, \text{ where } a, b, a', b' \leftarrow \psi_1. \quad (15)$$

| $\pm 3$ | $\pm 2$ | $\pm 1$ | 0    |
|---------|---------|---------|------|
| 1/128   | 1/32    | 23/128  | 9/16 |

Table 3: Probability distribution of  $c = ab + b'(a+a')$ 

| $\pm 2$ | $\pm 1$ | 0     |
|---------|---------|-------|
| 1/64    | 3/16    | 19/32 |

Table 4: Probability distribution of  $c' = ab + a'b'$ 

Additionally, for  $i \in [0, n/2 - 1]$ , the degree- $i$  coefficient of  $\mathbf{gr} + \mathbf{mf}'$  is the sum of  $n - 2i$  random variables  $c$  (as in Equation (15)), and  $2i$  independent random variables  $c'$  of the form:

$$c' = ba + b'a' \in \{0, \pm 1, \pm 2\} \text{ where } a, b, a', b' \leftarrow \psi_1. \quad (16)$$

Computing the probability distribution of this sum can be done via convolution (i.e., polynomial multiplication). Define the polynomial:

$$\rho_i(X) = \begin{cases} \sum_{j=-3n}^{3n} \rho_{i,j} X^j = \left( \sum_{j=-3}^3 \theta_j X^j \right)^n \text{ for } i = [n/2, n-1], \\ \sum_{j=-(3n-2i)}^{3n-2i} \rho_{i,j} X^j = \left( \sum_{j=-3}^3 \theta_j X^j \right)^{n-2i} \left( \sum_{j=-2}^2 \theta'_j X^j \right)^{2i} \text{ for } i = [0, n/2-1], \end{cases} \quad (17)$$

where  $\theta_j = \Pr[c = j]$  (see Table 3) and  $\theta'_j = \Pr[c' = j]$  (see Table 4). Let  $\rho_{i,j}$  be the probability that the degree- $i$  coefficient of  $\mathbf{gr} + \mathbf{mf}'$  is  $j$ . Then,  $\epsilon'_i$  can be computed as:

$$\epsilon'_i = \begin{cases} 2 \cdot \sum_{j=(q+3)/6}^{3n} \rho_{i,j} & \text{for } i \in [n/2, n-1], \\ 2 \cdot \sum_{j=(q+3)/6}^{3n-2i} \rho_{i,j} & \text{for } i \in [0, n/2-1], \end{cases}$$

using the symmetry  $\rho_{i,j} = \rho_{i,-j}$ . Substituting  $\epsilon'_i$  into Equation (14) yields the average-case correctness error  $\delta$  of  $\text{GenNTRU}[\psi_1^n]$ .

### 5.1.3 Injectivity

The injectivity of  $\text{GenNTRU}[\psi_1^n]$  can be easily shown as follows: if there exists an adversary that can yield two inputs  $(\mathbf{m}_1, \mathbf{r}_1)$  and  $(\mathbf{m}_2, \mathbf{r}_2)$  such that  $\text{Enc}(\mathbf{h}, \mathbf{m}_1; \mathbf{r}_1) = \text{Enc}(\mathbf{h}, \mathbf{m}_2; \mathbf{r}_2)$ , the equality indicates that  $(\mathbf{r}_1 - \mathbf{r}_2)\mathbf{h} + (\mathbf{m}_1 - \mathbf{m}_2) = 0$ , where  $\mathbf{r}_1 - \mathbf{r}_2$  and  $\mathbf{m}_1 - \mathbf{m}_2$  still have small coefficients of length, at most  $2\sqrt{n}$ . For a lattice set

$$\mathcal{L}_0^\perp := \{(\mathbf{v}, \mathbf{w}) \in R_q \times R_q : \mathbf{h}\mathbf{v} + \mathbf{w} = 0 \text{ (in } R_q)\},$$

$(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{m}_1 - \mathbf{m}_2)$  becomes an approximate shortest vector in  $\mathcal{L}_0^\perp$ . Thus, if the injectivity is broken against  $\text{GenNTRU}[\psi_1^n]$ , we can solve the approximate shortest vector problem (SVP) (of length at most  $2\sqrt{n}$ ) over  $\mathcal{L}_0^\perp$ . It is well-known [16] that the approximate SVP over  $\mathcal{L}_0^\perp$  is at least as hard as the  $\text{NTRU}_{n,q,\psi_1^n}$  problem (defined above). Hence, if the  $\text{NTRU}_{n,q,\psi_1^n}$  assumption holds, then the injectivity of  $\text{GenNTRU}[\psi_1^n]$  also holds.

### 5.1.4 Spreadness

**Lemma 5.2 (Spreadness).**  $\text{GenNTRU}[\psi_1^n]$  is  $n$ -spread.

*Proof.* For a fixed message  $\mathbf{m}$  and ciphertext  $\mathbf{c}$ , there exists at most one  $\mathbf{r}$  such that  $\mathbf{c} = \text{Enc}(\mathbf{h}, \mathbf{m}; \mathbf{r})$ . Suppose there exist  $\mathbf{r}_1$  and  $\mathbf{r}_2$  such that  $\mathbf{c} = \text{Enc}(\mathbf{h}, \mathbf{m}; \mathbf{r}_1) = \text{Enc}(\mathbf{h}, \mathbf{m}; \mathbf{r}_2)$ . Based on this assumption,

$hr_1 + m = hr_2 + m$  holds. By subtracting  $m$  and multiplying  $h^{-1}$  on both sides of the equation, we obtain  $r = r'$ . Therefore, there exists at most one  $r$  such that  $c = \text{Enc}(h, m; r)$ .

For fixed  $m$ , to maximize  $\Pr[\text{Enc}(h, m; r) = c]$ , we need to choose  $c$  such that  $c = \text{Enc}(h, m; r)$  for  $r = 0$ . Since there exists only one  $r$  such that  $c = \text{Enc}(h, m; r)$ , we have  $\Pr[\text{Enc}(h, m; r) = c] = 2^{-n}$ . Since this holds for any  $(pk, sk) \leftarrow \text{Gen}(1^\lambda)$  and  $m \in \mathcal{M}$ ,  $\text{GenNTRU}[\psi_1^n]$  is  $n$ -spread.  $\square$

### 5.1.5 Randomness-Recoverability

**Lemma 5.3.**  $\text{GenNTRU}[\psi_1^n]$  is randomness recoverable.

*Proof.* Suppose  $r = \text{RRec}(h, m, c) = (c - m)h^{-1} \in \mathcal{R} = \{-1, 0, 1\}$  for  $m \in \mathcal{M} = \{-1, 0, 1\}$  and  $c \in \mathcal{C} = R_q$ . Then, multiplying  $h$  and then adding  $m$  to both sides of  $r = (c - m)h^{-1}$  leads to  $c = hr + m = \text{Enc}(h, m; r)$ .  $\square$

### 5.1.6 Message-Recoverability

**Lemma 5.4.**  $\text{GenNTRU}[\psi_1^n]$  is AC-MR.

*Proof.* Suppose  $m = \text{MRec}(h, r, c) = c - hr \in \mathcal{M} = \{-1, 0, 1\}^n$  for  $r \in \mathcal{R} = \{-1, 0, 1\}^n$  and  $c \in \mathcal{C} = R_q$ . Then, adding  $hr$  to both sides of  $m = c - hr$  leads to  $c = hr + m = \text{Enc}(h, m; r)$ .  $\square$

**Lemma 5.5.**  $\text{GenNTRU}[\psi_1^n]$  is VC-MR.

*Proof.* Suppose that  $c = \text{Enc}(h, m; r) = hr + m$  with  $m \in \mathcal{M} = \{-1, 0, 1\}^n$  and  $r \in \mathcal{R} = \{-1, 0, 1\}^n$ . When we apply  $\text{MRec}$  to  $(h, r, c)$ , we obtain  $\text{MRec}(h, r, c) = c - hr = (hr + m) - hr = m$ . Therefore, whenever  $c$  is a valid ciphertext of  $m$  under randomness  $r$ , the algorithm  $\text{MRec}$  recovers  $m$  exactly, as required by VC-MR. Hence  $\text{GenNTRU}[\psi_1^n]$  is VC-MR.  $\square$

### 5.1.7 Randomness-Uniqueness

**Lemma 5.6.**  $\text{GenNTRU}[\psi_1^n]$  is randomness unique.

*Proof.* Suppose  $c = \text{Enc}(h, m; r) = hr + m$  and  $c' = \text{Enc}(h, m; r') = hr' + m$  satisfy  $c = c'$ . Then,  $hr + m = hr' + m$ . Since  $h$  is invertible, we can conclude that  $r = r'$  by subtracting  $m$  and then multiplying both sides of the equation by  $h$ .  $\square$

## 5.2 CPA-NTRU+ (=PKE')

### 5.2.1 Instantiation of SOTP

We introduce an instantiation of  $\text{SOTP} = (\text{Encode}, \text{Decode})$ , where  $\text{Encode} : \mathcal{M}' \times \mathcal{U} \rightarrow \mathcal{M}$  and  $\text{Decode} : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}'$ , with  $\mathcal{M}' = \{0, 1\}^n$ ,  $\mathcal{U} = \{0, 1\}^{2n}$ , and  $\mathcal{M} = \{-1, 0, 1\}^n$ , along with distributions  $\psi_{\mathcal{U}} = U^{2n}$  and  $\psi_{\mathcal{M}} = \psi_1^n$  as shown in Figure 19, which is used for  $\text{ACWC}_2$ . We note that, following [28], the values of  $y + u_2$  generated by  $\text{Decode}$  should be checked to determine whether they are 0 or 1.

**Message-Hiding and Rigidity Properties of SOTP.** It is easily shown that SOTP is message-hiding because of the one-time pad property, particularly for part  $x \oplus u_1$ . That is, unless  $u_1$  is known, the message  $x \in \mathcal{M}'$  is unconditionally hidden from  $y \in \mathcal{M}$ . Similarly,  $x \oplus u_1$  becomes uniformly random over  $\{0, 1\}^n$ , regardless of the message distribution  $\psi_{\mathcal{M}'}$ , and thus the resulting  $y$  follows  $\psi_1^n$ . In addition, we can easily check that SOTP is perfectly rigid as long as  $y + u_2 \in \{0, 1\}^n$ .

|  |
|--|
| $\text{Encode}(x \in \mathcal{M}', u \leftarrow U^{2n})$   |
| 1: $u = (u_1, u_2) \in \{0, 1\}^n \times \{0, 1\}^n$<br>2: $y = (x \oplus u_1) - u_2 \in \{-1, 0, 1\}^n$<br>3: <b>return</b> $y$ |

|  |
|--|
| $\text{Decode}(y \in \mathcal{M}, u \in U^{2n})$   |
| 1: $u = (u_1, u_2) \in \{0, 1\}^n \times \{0, 1\}^n$<br>2: <b>if</b> $y + u_2 \notin \{0, 1\}^n$ , <b>return</b> $\perp$<br>3: $x = (y + u_2) \oplus u_1 \in \{0, 1\}^n$<br>4: <b>return</b> $x$ |

Figure 19: SOTP instantiation for NTRU+

### 5.2.2 CPA-NTRU+ (=PKE')

We obtain CPA-NTRU+ := ACWC<sub>2</sub> [GenNTRU[ $\psi_1^n$ ], SOTP, G] by applying ACWC<sub>2</sub> from Section 3 to GenNTRU[ $\psi_1^n$ ]. Because the underlying GenNTRU[ $\psi_1^n$ ] provides injectivity, AC-MR, VC-MR, and RR properties, Theorems 3.6 and 3.7 provide us with the IND-CPA security of the resulting CPA-NTRU+ in the classical and quantum random oracle models, respectively. Regarding the correctness error, Theorem 3.2 shows that the worst-case correctness error of CPA-NTRU+ and the average-case correctness error of GenNTRU[ $\psi_1^n$ ] differ by the amount of  $\Delta = \|\psi_{\mathcal{R}}\| \cdot (1 + \sqrt{(\ln |\mathcal{M}'| - \ln \|\psi_{\mathcal{R}}\|)/2})$ , where  $\psi_{\mathcal{R}}$  and  $\mathcal{M}'$  are specified by  $\psi_1^n$  and  $\{0, 1\}^n$ , respectively. For instance, when  $n = 768$ , we obtain about  $\Delta = 2^{-1083}$ .

|   |   |
|---|---|
| $\text{Gen}'(1^\lambda)$  | $\text{Enc}'(pk, m \in \{0, 1\}^n; R \leftarrow \{0, 1\}^{2n})$   |
| 1: $(pk, sk) := \text{GenNTRU}[\psi_1^n].\text{Gen}(1^\lambda)$<br>- <b>repeat</b><br>- $\mathbf{f}'$ , $\mathbf{g} \leftarrow \psi_1^n$<br>- $\mathbf{f} = 3\mathbf{f}' + 1$<br>- <b>until</b> $\mathbf{f}$ is invertible in $R_q$<br>- <b>repeat</b><br>- $\mathbf{g} \leftarrow \psi_1^n$<br>- <b>until</b> $\mathbf{g}$ is invertible in $R_q$<br>- $(pk, sk) = (\mathbf{h} = 3\mathbf{g}\mathbf{f}^{-1} \bmod q, \mathbf{f})$<br>2: <b>return</b> $(pk, sk)$ | 1: $\mathbf{r} \leftarrow \psi_1^n$ using the randomness $R$<br>2: $\mathbf{m} = \text{Encode}(m, G(\mathbf{r}))$<br>3: $\mathbf{c} = \text{GenNTRU}[\psi_1^n].\text{Enc}(pk, \mathbf{m}; \mathbf{r})$<br>- $\mathbf{c} = \mathbf{h}\mathbf{r} + \mathbf{m}$<br>4: <b>return</b> $\mathbf{c}$   |
|   | $\text{Dec}'(sk, \mathbf{c})$   |
|   | 1: $\mathbf{m} = \text{GenNTRU}[\psi_1^n].\text{Dec}(sk, \mathbf{c})$<br>- $\mathbf{m} = (\mathbf{c}\mathbf{f} \bmod q) \bmod 3$<br>2: $\mathbf{r} = \text{RRec}(pk, \mathbf{c}, \mathbf{m})$<br>- $\mathbf{r} = (\mathbf{c} - \mathbf{m})\mathbf{h}^{-1}$<br>3: $m = \text{Decode}(\mathbf{m}, G(\mathbf{r}))$<br>4: <b>if</b> $m = \perp$ or $\mathbf{r} \notin \{-1, 0, 1\}^n$ , <b>return</b> $\perp$<br>5: <b>return</b> $m$ |

Figure 20: CPA-NTRU+

**Spreadness Properties of CPA-NTRU+.** To achieve IND-CCA security of the KEM and PKE via  $\overline{\text{FO}}^\perp$  and  $\overline{\text{FO}}_{\text{PKE}}^\perp$ , we need to show the spreadness of CPA-NTRU+. The spreadness can be easily obtained by combining Lemma 3.8 with Lemma 5.2.

### 5.3 NTRU+

Finally, we achieve IND-CCA secure KEM by applying  $\overline{\text{FO}}^\perp$  to CPA-NTRU+. We denote such KEM by NTRU+ :=  $\overline{\text{FO}}^\perp[\text{CPA-NTRU+}, H]$ . Figure 21 shows the resultant NTRU+, which is the basis of our implementation in the next section. By combining Theorems 4.1, 4.2, and Theorem 4.3, we can achieve

IND-CCA security of NTRU+. As for the correctness error, NTRU+ preserves the worst-case correctness error of the underlying CPA-NTRU+.

|   |  |
|---|--|
| <b>Gen</b> ( $1^\lambda$ )  | <b>Decap</b> ( $sk, \mathbf{c}$ )                              |
| 1: <b>repeat</b>  |  |
| 2: $\mathbf{f}', \mathbf{g} \leftarrow \psi_1^n$                                    | 1: $\mathbf{m} = (\mathbf{c}\mathbf{f} \bmod q) \bmod 3$       |
| 3: $\mathbf{f} = 3\mathbf{f}' + 1$  | 2: $\mathbf{r} = (\mathbf{c} - \mathbf{m})\mathbf{h}^{-1}$     |
| 4: <b>until</b> $\mathbf{f}$ is invertible in $R_q$                                 | 3: $m = \text{Decode}(\mathbf{m}, \mathbf{G}(\mathbf{r}))$     |
| 5: <b>repeat</b>  | 4: $(R', K) = \mathbf{H}(m)$                                   |
| 6: $\mathbf{g} \leftarrow \psi_1^n$   | 5: $\mathbf{r}' \leftarrow \psi_1^n$ using the randomness $R'$ |
| 7: <b>until</b> $\mathbf{g}$ is invertible in $R_q$                                 | 6: <b>if</b> $m = \perp$ or $\mathbf{r} \neq \mathbf{r}'$      |
| 8: <b>return</b> $(pk, sk) = (\mathbf{h} = 3\mathbf{g}\mathbf{f}^{-1}, \mathbf{f})$ | 7: <b>return</b> $\perp$                                       |
| <b>Encap</b> ( $pk$ )   | 8: <b>else</b>   |
| 1: $m \leftarrow \{0, 1\}^n$  | 9: <b>return</b> $K$   |
| 2: $(R, K) = \mathbf{H}(m)$   |  |
| 3: $\mathbf{r} \leftarrow \psi_1^n$ using the randomness $R$                        |  |
| 4: $\mathbf{m} = \text{Encode}(m, \mathbf{G}(\mathbf{r}))$                          |  |
| 5: $\mathbf{c} = \mathbf{h}\mathbf{r} + \mathbf{m}$                                 |  |
| 6: <b>return</b> $(\mathbf{c}, K)$  |  |

Figure 21: NTRU+

## 6 Algorithm Specification

### 6.1 Auxiliary Functions

**Bit Ordering Functions.** To convert between byte arrays and bit arrays, we introduce the BytesToBits and BitsToBytes functions in Algorithms 1 and 2. These functions form an inverse pair, so applying one after the other recovers the original input. They are used later in Algorithms 3, 4, and 5.

---

#### Algorithm 1: BytesToBits

---

**Require:** Byte array  $B = (b_0, b_1, \dots, b_{n/8-1}) \in \mathcal{B}^{n/8}$   
**Ensure:** Bit array  $f = (f_0, \dots, f_{n-1}) \in \{0, 1\}^n$

- 1: **for**  $i$  from 0 to  $n/8 - 1$  **do**
- 2:    $t = b_i$
- 3:   **for**  $j$  from 0 to 7 **do**
- 4:      $f_{8i+j} = t \& 1$
- 5:      $t = t \gg 1$
- 6: **return**  $f = (f_0, \dots, f_{n-1})$

---



---

#### Algorithm 2: BitsToBytes

---

**Require:** Bit array  $f = (f_0, \dots, f_{n-1}) \in \{0, 1\}^n$   
**Ensure:** Byte array  $B = (b_0, b_1, \dots, b_{n/8-1}) \in \mathcal{B}^{n/8}$

- 1: **for**  $i$  from 0 to  $n/8 - 1$  **do**
- 2:    $b_i = 0$
- 3:   **for**  $j$  from 0 to 7 **do**
- 4:      $b_i = b_i + f_{8i+j} \times 2^j$
- 5: **return**  $(b_0, \dots, b_{n/8-1})$

---

**Sampling from a Binomial Distribution.** To sample the coefficients of a polynomial from the centered binomial distribution with  $\eta = 1$ , we introduce the CBD<sub>1</sub> function in Algorithm 3. To determine the bit ordering of the input bytes, this function uses the BytesToBits function defined in Algorithm 1.

---

#### Algorithm 3: CBD<sub>1</sub>

---

**Require:** Byte array  $B = (b_0, b_1, \dots, b_{n/4-1})$   
**Ensure:** Polynomial  $\mathbf{f} \in R_q$

- 1:  $(\beta_0, \dots, \beta_{n-1}) := \text{BytesToBits}((b_0, \dots, b_{n/8-1}))$
- 2:  $(\beta_n, \dots, \beta_{2n-1}) := \text{BytesToBits}((b_{n/8}, \dots, b_{n/4-1}))$
- 3: **for**  $i$  from 0 to  $n - 1$  **do**
- 4:    $f_i := \beta_i - \beta_{i+n}$
- 5: **return**  $\mathbf{f} = f_0 + f_1 x + f_2 x^2 + \dots + f_{n-1} x^{n-1}$

---

**Semi-generalized One-Time Pad (SOTP).** We define SOTP = (Encode, Decode) in Algorithms 4 and 5, respectively. The Encode function is identical to CBD<sub>1</sub>, except that it XORs the first half of the random bytes with the message before applying the centered binomial sampling. Accordingly, Encode also uses the BytesToBits (Algorithm 1), as in CBD<sub>1</sub>. The corresponding Decode function, given in Algorithm 5, acts as the inverse of Encode and uses BitsToBytes (Algorithm 2) to recover the original byte array.

---

**Algorithm 4:** Encode

---

**Require:** Message Byte array  $m = (m_0, m_1, \dots, m_{31})$   
**Require:** Byte array  $B = (b_0, b_1, \dots, b_{n/4-1})$   
**Ensure:** Polynomial  $\mathbf{f} \in R_q$

- 1:  $(\beta_0, \dots, \beta_{n-1}) := \text{BytesToBits}((b_0, \dots, b_{n/8-1}))$
- 2:  $(\beta_n, \dots, \beta_{2n-1}) := \text{BytesToBits}((b_{n/8}, \dots, b_{n/4-1}))$
- 3:  $(m_0, \dots, m_{n-1}) := \text{BytesToBits}(m)$
- 4: **for**  $i$  from 0 to  $n-1$  **do**
- 5:    $f_i := (m_i \oplus \beta_i) - \beta_{i+n}$
- 6: **return**  $\mathbf{f} = f_0 + f_1x + f_2x^2 + \dots + f_{n-1}x^{n-1}$

---

---

**Algorithm 5:** Decode

---

**Require:** Polynomial  $\mathbf{f} \in R_q$   
**Require:** Byte array  $B = (b_0, b_1, \dots, b_{n/4-1})$   
**Ensure:** Message Byte array  $m = (m_0, m_1, \dots, m_{31})$

- 1:  $(\beta_0, \dots, \beta_{n-1}) := \text{BytesToBits}((b_0, \dots, b_{n/8-1}))$
- 2:  $(\beta_n, \dots, \beta_{2n-1}) := \text{BytesToBits}((b_{n/8}, \dots, b_{n/4-1}))$
- 3: **for**  $i$  from 0 to  $n-1$  **do**
- 4:   **if**  $f_i + \beta_{i+n} \notin \{0, 1\}$ , **return**  $\perp$  ; //Refer to line 8 in Algorithm 13
- 5:    $m_i := ((f_i + \beta_{i+n}) \& 1) \oplus \beta_i$   
 $m = \text{BitsToBytes}((m_0, \dots, m_{n-1}))$
- 6: **return**  $m$

---

**Encoding and Decoding.** To convert a polynomial in  $R_q$  to and from its  $3n/2$ -byte representation, we introduce the  $\text{Encode}_q$  and  $\text{Decode}_q$  functions in Algorithms 6 and 7. The  $\text{Encode}_q$  function assumes that each coefficient of the input polynomial lies in  $\{0, \dots, q-1\}$  before packing it into the byte array. The  $\text{Decode}_q$  function performs the inverse transformation by recovering the coefficients from the byte array. Together, the two functions form an inverse pair, so applying one after the other recovers the original polynomial.

---

**Algorithm 6:**  $\text{Encode}_q$ 

---

**Require:** Polynomial  $\mathbf{f} = (f_0, \dots, f_{n-1}) \in R_q$ , with each  $f_i \in \{0, \dots, q-1\}$   
**Ensure:** Byte array  $B = (b_0, \dots, b_{3n/2-1})$

- 1: **for**  $i$  from 0 to  $n/2-1$  **do**
- 2:    $t_0 = f_{2i}$
- 3:    $t_1 = f_{2i+1}$
- 4:    $b_{3i} = t_0$
- 5:    $b_{3i+1} = (t_0 \gg 8) + (t_1 \ll 4)$
- 6:    $b_{3i+2} = t_1 \gg 4$
- 7: **return**  $B$

---

---

**Algorithm 7:**  $\text{Decode}_q$ 

---

**Require:** Byte array  $B = (b_0, \dots, b_{3n/2-1})$   
**Ensure:** Polynomial  $\mathbf{f} = (f_0, \dots, f_{n-1}) \in R_q$ , with each  $f_i \in \{0, \dots, q-1\}$

- 1: **for**  $i$  from 0 to  $n/2 - 1$  **do**
- 2:    $t_0 = b_{3i}$
- 3:    $t_1 = b_{3i+1}$
- 4:    $t_2 = b_{3i+2}$
- 5:    $f_{2i} = t_0 \mid ((t_1 \& 0x0f) \ll 8)$
- 6:    $f_{2i+1} = ((t_1 \gg 4) \& 0x0f) \mid (t_2 \ll 4)$
- 7: **return**  $\mathbf{f}$

---

**Symmetric Primitives.** The scheme uses three distinct hash functions, denoted by  $\mathbf{F}$ ,  $\mathbf{G}$ , and  $\mathbf{H}$ . Each function is instantiated with SHAKE-256, as shown in Algorithms 8, 9, and 10. We also use SHAKE-256 as an extendable-output function (XOF) when sampling the polynomials  $\mathbf{f}'$  and  $\mathbf{g}'$  in Algorithm 11.

---

**Algorithm 8:**  $\mathbf{F}$ 

---

**Require:** Byte array  $m = (m_0, m_1, \dots, m_{3n/2-1})$   
**Ensure:** Byte array  $B = (b_0, b_1, \dots, b_{31})$

- 1:  $(b_0, \dots, b_{31}) := \text{SHAKE-256}(0x00 \parallel m)$
- 2: **return**  $(b_0, \dots, b_{31})$

---

---

**Algorithm 9:**  $\mathbf{G}$ 

---

**Require:** Byte array  $m = (m_0, m_1, \dots, m_{n/8-1})$   
**Ensure:** Byte array  $B = (b_0, b_1, \dots, b_{n/4-1})$

- 1:  $(b_0, \dots, b_{n/4-1}) := \text{SHAKE-256}(0x01 \parallel m, n/4)$
- 2: **return**  $(b_0, \dots, b_{n/4-1})$

---

---

**Algorithm 10:**  $\mathbf{H}$ 

---

**Require:** Byte array  $m = (m_0, m_1, \dots, m_{n/8+31})$   
**Ensure:** Byte array  $B = (b_0, b_1, \dots, b_{n/4+31})$

- 1:  $(b_0, \dots, b_{n/4+31}) := \text{SHAKE-256}(0x02 \parallel m, n/4 + 32)$
- 2: **return**  $(b_0, \dots, b_{n/4+31})$

---

## 6.2 Number Theoretic Transform

**Polynomial Rings and Number Theoretic Transform** Throughout this specification, we consider the quotient rings  $R = \mathbb{Z}[x]/\langle \Phi_{3n}(x) \rangle$  and  $R_q = \mathbb{Z}_q[x]/\langle \Phi_{3n}(x) \rangle$ , where  $\Phi_{3n}(x) = x^n - x^{n/2} + 1$  denotes the  $3n$ -th cyclotomic polynomial of degree  $n = 2^{a+1}3^b$  for  $a, b \in \mathbb{N} \cup \{0\}$ . To optimize polynomial multiplication, the Number Theoretic Transform (NTT) is used to establish a ring isomorphism based on the Generalized Chinese Remainder Theorem (GCRT):

$$R_q \cong \prod_{i=0}^{n/d-1} \mathbb{Z}_q[x]/\langle x^d - \zeta^{\text{index}[i]} \rangle, \quad d \in \{3, 4\}.$$

Here,  $\zeta$  is a primitive  $\ell$ -th root of unity modulo  $q$  with  $\ell = 3n/d$ , where the values  $(d, \zeta, \ell)$  are defined in Table 5, and the array `index` is defined in Figure 22. This decomposition ensures that high-degree multiplication is reduced to  $n/d$  independent multiplications in smaller component rings. Under this decomposition, the forward transform NTT can be written as

$$\begin{aligned} \hat{f} = \text{NTT}(f) &= \left( f \bmod (x^d - \zeta^{\text{index}[0]}), \dots, f \bmod (x^d - \zeta^{\text{index}[n/d-1]}) \right) \\ &= \underbrace{(\hat{f}_0, \dots, \hat{f}_{d-1})}_{\mathbb{Z}_q[x]/\langle x^d - \zeta^{\text{index}[0]} \rangle} \parallel \dots \parallel \underbrace{(\hat{f}_{n-d}, \dots, \hat{f}_{n-1})}_{\mathbb{Z}_q[x]/\langle x^d - \zeta^{\text{index}[n/d-1]} \rangle}, \end{aligned}$$

and the inverse transform  $\text{NTT}^{-1}$  is defined as its inverse.

Concretely, NTT is performed through three different sequential stages of butterfly layers, with iterations determined by the parameter set in Table 5:

1. **Initial Radix-2 Layer:** This layer decomposes the initial cyclotomic ring  $R_q$  into two sub-rings:

$$R_q = \mathbb{Z}_q[x]/\langle x^n - x^{n/2} + 1 \rangle \cong \mathbb{Z}_q[x]/\langle x^{n/2} - \zeta^{\ell/6} \rangle \times \mathbb{Z}_q[x]/\langle x^{n/2} - \zeta^{5\ell/6} \rangle.$$

This factorization provides the initial split for the recursive transform [31].

2. **Standard Radix-3 Layer:** This layer decomposes the quotient ring as

$$\mathbb{Z}_q[x]/\langle x^{3m} - \alpha^3 \rangle \cong \prod_{j=0}^2 \mathbb{Z}_q[x]/\langle x^m - \alpha\omega^j \rangle,$$

where  $\omega = \zeta^{\ell/3}$  is a primitive third root of unity (see [18] or Appendix B).

3. **Standard Radix-2 Layer:** This layer decomposes the quotient ring as

$$\mathbb{Z}_q[x]/\langle x^{2m} - \psi^2 \rangle \cong \mathbb{Z}_q[x]/\langle x^m - \psi \rangle \times \mathbb{Z}_q[x]/\langle x^m + \psi \rangle.$$

This decomposition can be equivalently written as  $\mathbb{Z}_q[x]/\langle x^m - \psi \rangle \times \mathbb{Z}_q[x]/\langle x^m - \psi\zeta^{\ell/2} \rangle$  because  $\zeta^{\ell/2} \equiv -1 \pmod{q}$ . Since  $\ell/2$  is even, this bisection is well-defined provided that  $\psi$  is a quadratic residue. For the chosen parameter sets, this bisection continues until  $m = d \in \{3, 4\}$ . Note that the Radix-3 Layer is executed prior to the Radix-2 Layer to avoid the need for an additional pre-computation table for the base multiplication.

By combining these sequential layers, the initial ring is decomposed into a collection of smaller component rings, ordered according to the `index` array. To make this construction explicit, we describe how the `index` array is generated by following the sequence of butterfly layers defined above. Starting from the initial root values, the corresponding layer operations are applied recursively according to the prescribed schedule in Table 5, with each layer expanding all current leaf nodes before proceeding to the next layer.

- **Initial Radix-2 Layer:** Initializes two independent trees with the root values  $\{\ell/6, 5\ell/6\}$ .
- **Standard Radix-3 Layer:** For each application of the radix-3 layer, every current leaf node with value  $\psi$  is expanded into three ordered child nodes  $\{\psi/3, (\psi + \ell)/3, (\psi + 2\ell)/3\}$ .
- **Standard Radix-2 Layer:** For each application of the radix-2 layer, every current leaf node with value  $\psi$  is expanded into two ordered child nodes  $\{\psi/2, (\psi + \ell)/2\}$ .

**Multiplication in the NTT Domain.** After transforming a polynomial in  $R_q$  into its components in the product rings, multiplication is performed independently in each ring  $\mathbb{Z}_q[x]/\langle x^d - \zeta \rangle$ . Let  $a(x) = \sum_{j=0}^{d-1} a_j x^j$  and  $b(x) = \sum_{j=0}^{d-1} b_j x^j$  be polynomials in this ring.

For  $d = 2$ , the product  $c(x) = a(x)b(x)$  is

$$c(x) = a(x)b(x) = (a_0 b_0 + a_1 b_1 \zeta) + (a_0 b_1 + a_1 b_0)x,$$

which can be written in matrix form as

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} a_0 & a_1 \zeta \\ a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

For  $d = 3$ , the product  $c(x) = a(x)b(x)$  becomes:

$$c(x) = a(x)b(x) = (a_0 b_0 + (a_2 b_1 + a_1 b_2) \zeta) + (a_1 b_0 + a_0 b_1 + a_2 b_2 \zeta)x + (a_2 b_0 + a_1 b_1 + a_0 b_2)x^2,$$

which corresponds to the matrix representation

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_0 & a_2 \zeta & a_1 \zeta \\ a_1 & a_0 & a_2 \zeta \\ a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}.$$

For  $d = 4$ , the product  $c(x) = a(x)b(x)$  is

$$\begin{aligned} c(x) = a(x)b(x) = & (a_0 b_0 + (a_1 b_3 + a_2 b_2 + a_3 b_1) \zeta) + (a_0 b_1 + a_1 b_0 + (a_2 b_3 + a_3 b_2) \zeta)x \\ & + (a_0 b_2 + a_1 b_1 + a_2 b_0 + a_3 b_3 \zeta)x^2 + (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0)x^3, \end{aligned}$$

with the corresponding matrix form

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_0 & a_3 \zeta & a_2 \zeta & a_1 \zeta \\ a_1 & a_0 & a_3 \zeta & a_2 \zeta \\ a_2 & a_1 & a_0 & a_3 \zeta \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

**Inversion in the NTT Domain.** In the NTT domain, inversion is performed independently in each component ring  $\mathbb{Z}_q[x]/\langle x^d - \zeta \rangle$ , analogous to multiplication. Let  $a(x) = \sum_{j=0}^{d-1} a_j x^j$  and write its inverse as  $b(x) = \sum_{j=0}^{d-1} b_j x^j$ .

For  $d = 2$ , the inverse  $b(x)$  is obtained from

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} a_0 & a_1 \zeta \\ a_1 & a_0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{a_0^2 - a_1^2 \zeta} \begin{bmatrix} a_0 & -a_1 \zeta \\ -a_1 & a_0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{a_0^2 - a_1^2 \zeta} \begin{bmatrix} a_0 \\ -a_1 \end{bmatrix}.$$

For  $d = 3$ , we similarly compute

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_0 & a_2 \zeta & a_1 \zeta \\ a_1 & a_0 & a_2 \zeta \\ a_2 & a_1 & a_0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = d^{-1} \begin{bmatrix} a'_0 \\ a'_1 \\ a'_2 \end{bmatrix},$$

where  $a'_0 = a_0^2 - \zeta a_1 a_2$ ,  $a'_1 = \zeta a_2^2 - a_0 a_1$ ,  $a'_2 = a_1^2 - a_0 a_2$ , and  $d = a_0 a'_0 + \zeta(a_1 a'_2 + a_2 a'_1)$ .

For  $d = 4$ , direct matrix inversion is less efficient. Following [35], we reduce inversion in  $\mathbb{Z}_q[x]/\langle x^4 - \zeta \rangle$  to inversion in  $\mathbb{Z}_q[z]/\langle z^2 - \zeta \rangle$ , where  $z = x^2$ . We rewrite

$$a(x) = \tilde{a}_0(z) + \tilde{a}_1(z)x, \quad \text{where } \tilde{a}_0(z) = a_0 + a_2 z, \tilde{a}_1(z) = a_1 + a_3 z.$$

For  $b(x) = \tilde{b}_0(z) + \tilde{b}_1(z)x$ , the product becomes

$$\begin{aligned} c(x) = a(x)b(x) &= (\tilde{a}_0(z) + \tilde{a}_1(z)x) \cdot (\tilde{b}_0(z) + \tilde{b}_1(z)x) \\ &= \tilde{a}_0(z)\tilde{b}_0(z) + (\tilde{a}_0(z)\tilde{b}_1(z) + \tilde{a}_1(z)\tilde{b}_0(z))x + \tilde{a}_1(z)\tilde{b}_1(z)x^2 \\ &= (\tilde{a}_0(z)\tilde{b}_0(z) + \tilde{a}_1(z)\tilde{b}_1(z)z) + (\tilde{a}_0(z)\tilde{b}_1(z) + \tilde{a}_1(z)\tilde{b}_0(z))x, \end{aligned}$$

which corresponds to the matrix product

$$\begin{bmatrix} \tilde{c}_0(z) \\ \tilde{c}_1(z) \end{bmatrix} = \begin{bmatrix} \tilde{a}_0(z) & \tilde{a}_1(z)z \\ \tilde{a}_1(z) & \tilde{a}_0(z) \end{bmatrix} \begin{bmatrix} \tilde{b}_0(z) \\ \tilde{b}_1(z) \end{bmatrix}.$$

To find the inverse  $b(x) = \tilde{b}_0(z) + \tilde{b}_1(z)x$ , we use:

$$\begin{aligned} \begin{bmatrix} \tilde{b}_0(z) \\ \tilde{b}_1(z) \end{bmatrix} &= \begin{bmatrix} \tilde{a}_0(z) & \tilde{a}_1(z)z \\ \tilde{a}_1(z) & \tilde{a}_0(z) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\tilde{a}_0^2(z) - \tilde{a}_1^2(z)z} \begin{bmatrix} \tilde{a}_0(z) & -\tilde{a}_1(z)z \\ -\tilde{a}_1(z) & \tilde{a}_0(z) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{\tilde{a}_0^2(z) - \tilde{a}_1^2(z)z} \begin{bmatrix} \tilde{a}_0(z) \\ -\tilde{a}_1(z) \end{bmatrix} \in \mathbb{Z}_q[z]/\langle z^2 - \zeta \rangle^{1 \times 2}. \end{aligned}$$

The inverse of  $\tilde{a}_0^2(z) - \tilde{a}_1^2(z)z$  in  $\mathbb{Z}_q[z]/\langle z^2 - \zeta \rangle$  can be computed using the  $d = 2$  case, and once the inversion is completed in that ring, the final result is obtained by substituting  $z = x^2$ .

In all cases, the multiplicative inverse modulo  $q$  must be computed. To reduce the risk of side-channel leakage, we use Fermat's Little Theorem instead of the extended Euclidean algorithm: if  $a \neq 0$  and is coprime with  $q$ , then  $a^{q-1} \equiv 1 \pmod{q}$ , and thus the inverse of  $a$  can be obtained as  $a^{q-2} \pmod{q}$ . Note that  $a = 0$  is never invertible in  $\mathbb{Z}_q$ , and such values are excluded by construction.

| $n$  | $q$  | Initial Radix-2 | Standard Radix-3 | Standard Radix-2 | $d$ | $\zeta$ | $\ell$ |
|------|------|-----------------|------------------|------------------|-----|---------|--------|
| 768  | 3457 | 1               | 1                | 5                | 4   | 22      | 576    |
| 864  | 3457 | 1               | 2                | 4                | 3   | 9       | 864    |
| 1152 | 3457 | 1               | 2                | 4                | 4   | 9       | 864    |

Table 5: Layer configurations for the NTT

- NTRU+768

```
index[192] = {
    1, 289, 145, 433, 73, 361, 217, 505, 37, 325, 181, 469, 109, 397, 253, 541,
    19, 307, 163, 451, 91, 379, 235, 523, 55, 343, 199, 487, 127, 415, 271, 559,
    7, 295, 151, 439, 79, 367, 223, 511, 43, 331, 187, 475, 115, 403, 259, 547,
    25, 313, 169, 457, 97, 385, 241, 529, 61, 349, 205, 493, 133, 421, 277, 565,
    13, 301, 157, 445, 85, 373, 229, 517, 49, 337, 193, 481, 121, 409, 265, 553,
    31, 319, 175, 463, 103, 391, 247, 535, 67, 355, 211, 499, 139, 427, 283, 571,
    5, 293, 149, 437, 77, 365, 221, 509, 41, 329, 185, 473, 113, 401, 257, 545,
    23, 311, 167, 455, 95, 383, 239, 527, 59, 347, 203, 491, 131, 419, 275, 563,
    11, 299, 155, 443, 83, 371, 227, 515, 47, 335, 191, 479, 119, 407, 263, 551,
    29, 317, 173, 461, 101, 389, 245, 533, 65, 353, 209, 497, 137, 425, 281, 569,
    17, 305, 161, 449, 89, 377, 233, 521, 53, 341, 197, 485, 125, 413, 269, 557,
    35, 323, 179, 467, 107, 395, 251, 539, 71, 359, 215, 503, 143, 431, 287, 575
};
```

- NTRU+864 and NTRU+1152

```
index[288] = {
    1, 433, 217, 649, 109, 541, 325, 757, 55, 487, 271, 703, 163, 595, 379, 811,
    19, 451, 235, 667, 127, 559, 343, 775, 73, 505, 289, 721, 181, 613, 397, 829,
    37, 469, 253, 685, 145, 577, 361, 793, 91, 523, 307, 739, 199, 631, 415, 847,
    7, 439, 223, 655, 115, 547, 331, 763, 61, 493, 277, 709, 169, 601, 385, 817,
    25, 457, 241, 673, 133, 565, 349, 781, 79, 511, 295, 727, 187, 619, 403, 835,
    43, 475, 259, 691, 151, 583, 367, 799, 97, 529, 313, 745, 205, 637, 421, 853,
    13, 445, 229, 661, 121, 553, 337, 769, 67, 499, 283, 715, 175, 607, 391, 823,
    31, 463, 247, 679, 139, 571, 355, 787, 85, 517, 301, 733, 193, 625, 409, 841,
    49, 481, 265, 697, 157, 589, 373, 805, 103, 535, 319, 751, 211, 643, 427, 859,
    5, 437, 221, 653, 113, 545, 329, 761, 59, 491, 275, 707, 167, 599, 383, 815,
    23, 455, 239, 671, 131, 563, 347, 779, 77, 509, 293, 725, 185, 617, 401, 833,
    41, 473, 257, 689, 149, 581, 365, 797, 95, 527, 311, 743, 203, 635, 419, 851,
    11, 443, 227, 659, 119, 551, 335, 767, 65, 497, 281, 713, 173, 605, 389, 821,
    29, 461, 245, 677, 137, 569, 353, 785, 83, 515, 299, 731, 191, 623, 407, 839,
    47, 479, 263, 695, 155, 587, 371, 803, 101, 533, 317, 749, 209, 641, 425, 857,
    17, 449, 233, 665, 125, 557, 341, 773, 71, 503, 287, 719, 179, 611, 395, 827,
    35, 467, 251, 683, 143, 575, 359, 791, 89, 521, 305, 737, 197, 629, 413, 845,
    53, 485, 269, 701, 161, 593, 377, 809, 107, 539, 323, 755, 215, 647, 431, 863
};
```

Figure 22: Index for the NTT

### 6.3 Specification of NTRU+

#### 6.3.1 NTRU+

We specify the key encapsulation mechanism NTRU+. In contrast to the construction in Section 5.3, this version instantiates a variant of  $\overline{\text{FO}}^\perp$  in which the public key is hashed into the key-derivation input, following the public-key-dependent FO transformation first introduced in KYBER [34] to enhance robustness in multi-user settings. Algorithms 11, 12, and 13 specify the key generation, encapsulation, and decapsulation procedures of NTRU+.

---

**Algorithm 11:**  $\text{Gen}(1^\lambda)$ : key generation

---

**Ensure:** Public key  $pk \in \mathcal{B}^{\lceil \log_2 q \rceil \cdot n/8}$   
**Ensure:** Secret key  $sk \in \mathcal{B}^{\lceil \log_2 q \rceil \cdot n/4 + 32}$

- 1: **repeat**
- 2:    $d \leftarrow \mathcal{B}^{32}$
- 3:    $f := \text{SHAKE-256}(d, n/4)$
- 4:    $\mathbf{f}' := \text{CBD}_1(f)$
- 5:    $\mathbf{f} := 3\mathbf{f}' + 1$
- 6:    $\hat{\mathbf{f}} := \text{NTT}(\mathbf{f})$
- 7: **until**  $\mathbf{f}$  is invertible in  $R_q$
- 8: **repeat**
- 9:    $d \leftarrow \mathcal{B}^{32}$
- 10:    $g := \text{SHAKE-256}(d, n/4)$
- 11:    $\mathbf{g}' := \text{CBD}_1(g)$
- 12:    $\mathbf{g} := 3\mathbf{g}'$
- 13:    $\hat{\mathbf{g}} := \text{NTT}(\mathbf{g})$
- 14: **until**  $\mathbf{g}$  is invertible in  $R_q$
- 15:  $\hat{\mathbf{h}} := \hat{\mathbf{g}} \circ \hat{\mathbf{f}}^{-1}$
- 16:  $pk := \text{Encode}_q(\hat{\mathbf{h}})$
- 17:  $sk := \text{Encode}_q(\hat{\mathbf{f}}) \parallel \text{Encode}_q(\hat{\mathbf{h}}^{-1}) \parallel \mathsf{F}(pk)$
- 18: **return**  $(pk, sk)$

---

**Algorithm 12:**  $\text{Encap}(pk)$ : encapsulation

---

**Require:** Public key  $pk \in \mathcal{B}^{\lceil \log_2 q \rceil \cdot n/8}$   
**Ensure:** Ciphertext  $c \in \mathcal{B}^{\lceil \log_2 q \rceil \cdot n/8}$

- 1:  $m \leftarrow \mathcal{B}^{n/8}$
- 2:  $(K, r) := \mathsf{H}(m, \mathsf{F}(pk))$
- 3:  $\mathbf{r} := \text{CBD}_1(r)$
- 4:  $\hat{\mathbf{r}} := \text{NTT}(\mathbf{r})$
- 5:  $\mathbf{m} := \text{Encode}(m, \mathsf{G}(\text{Encode}_q(\hat{\mathbf{r}})))$
- 6:  $\hat{\mathbf{m}} := \text{NTT}(\mathbf{m})$
- 7:  $\hat{\mathbf{h}} := \text{Decode}_q(pk)$
- 8:  $\hat{\mathbf{c}} := \hat{\mathbf{h}} \circ \hat{\mathbf{r}} + \hat{\mathbf{m}}$
- 9:  $c := \text{Encode}_q(\hat{\mathbf{c}})$
- 10: **return**  $(c, K)$

---

---

**Algorithm 13:** Decap( $sk, c$ ): decapsulation

---

**Require:** Secret key  $sk \in \mathcal{B}^{\lceil \log_2 q \rceil \cdot n/4 + 32}$

**Require:** Ciphertext  $c \in \mathcal{B}^{\lceil \log_2 q \rceil \cdot n/8}$

**Ensure:** Shared key  $K \in \mathcal{B}^{32}$

- 1: Parse  $sk = (sk_1, sk_2, sk_3) \in \mathcal{B}^{\lceil \log_2 q \rceil \cdot n/8} \times \mathcal{B}^{\lceil \log_2 q \rceil \cdot n/8} \times \mathcal{B}^{32}$
  - 2:  $\hat{\mathbf{f}} := \text{Decode}_q(sk_1)$
  - 3:  $\hat{\mathbf{c}} := \text{Decode}_q(c)$
  - 4:  $\mathbf{m} := \text{NTT}^{-1}(\hat{\mathbf{c}} \circ \hat{\mathbf{f}}) \bmod \pm 3$
  - 5:  $\hat{\mathbf{m}} := \text{NTT}(\mathbf{m})$
  - 6:  $\hat{\mathbf{h}}^{-1} := \text{Decode}_q(sk_2)$
  - 7:  $\hat{\mathbf{r}} := (\hat{\mathbf{c}} - \hat{\mathbf{m}}) \circ \hat{\mathbf{h}}^{-1}$  //randomness recovery (RRec)
  - 8:  $m' := \text{Decode}(\mathbf{m}, G(\text{Encode}_q(\hat{\mathbf{r}})))$  //Check  $m' = \perp$  in line 12
  - 9:  $(K, r') := H(m', sk_3)$
  - 10:  $\mathbf{r}' := \text{CBD}_1(r')$
  - 11:  $\hat{\mathbf{r}}' := \text{NTT}(\mathbf{r}')$
  - 12: **if**  $m' = \perp$  or  $\hat{\mathbf{r}} \neq \hat{\mathbf{r}}'$ , **return**  $\perp$  //Check if  $m' = \perp$  or  $\mathbf{r}' \notin R_q$
  - 13: **else**, **return**  $K$
-

## 7 Parameters and Security Analysis

We define three parameter sets for NTRU+, which are listed in Table 7. We denote them by NTRU+{768, 864, and 1152}, respectively, depending on the degree  $n \in \{768, 864, 1152\}$  of the polynomial  $x^n - x^{n/2} + 1$ . In all parameter sets the modulus is fixed to  $q = 3457$ , and the coefficients of  $\mathbf{m}$  and  $\mathbf{r}$  are sampled according to the distribution  $\psi_1^n$  (i.e.,  $\psi_{\mathcal{R}} = \psi_{\mathcal{M}} = \psi_1^n$ ). For each tuple  $(n, q, \psi_1^n, \mathcal{M}' = \{0, 1\}^n)$ , the worst-case correctness error  $\delta'$  is calculated as  $\delta' = \delta + \Delta$ , where  $\delta$  is the average-case correctness error of GenNTRU $[\psi_1^n]$  and  $\Delta = \|\psi_{\mathcal{R}}\| \cdot (1 + \sqrt{(\ln |\mathcal{M}'| - \ln \|\psi_{\mathcal{R}}\|)/2})$  as given in Theorem 3.2. Since  $\Delta$  is negligible for all parameter sets, the worst-case correctness error of NTRU+ is essentially identical to the average-case correctness error of the corresponding GenNTRU $[\psi_1^n]$ .

To estimate the concrete security level of NTRU+, we analyze the hardness of the RLWE $_{n, q, \psi_1^n}$  and NTRU $_{n, q, \psi_1^n}$  problems for each parameter set. For the RLWE problem, we employ the Lattice estimator [1]<sup>5</sup>, which evaluates the best-known lattice attacks using the BKZ reduction algorithm [11], including the primal [2] and dual [29] attacks. For the NTRU problem, we use the NTRU estimator provided by the finalist NTRU submission [10], which incorporates both the primal attack and Howgrave–Graham’s hybrid attack [21] over the NTRU lattice. The primal attack over the NTRU lattice is essentially the same as the attack using the BKZ algorithm, and Howgrave–Graham’s hybrid attack is also based on the BKZ algorithm combined with Odlyzko’s Meet-in-the-Middle (MitM) attack [24] on a (reduced) sub-lattice. As a result, the concrete security level of the NTRU problem closely matches that of the RLWE problem. Table 6 summarizes the resulting security levels for each NTRU+ parameter set. For the BKZ cost model, we use  $2^{0.292\beta}$  [5] in the classical setting and  $2^{0.257\beta}$  [9] in the quantum setting.

Recently, Lee *et al.* [27] proposed a combinatorial attack that improves upon May’s Meet-LWE attack [32] and analyzed the concrete security level of NTRU+. Their analysis shows that the security of NTRU+ against this combinatorial attack does not fall below the levels predicted by the above Lattice and NTRU estimators.

| Scheme    | classical |      | quantum |      |
|-----------|-----------|------|---------|------|
|           | LWE       | NTRU | LWE     | NTRU |
| NTRU+768  | 156       | 164  | 139     | 144  |
| NTRU+864  | 179       | 189  | 160     | 166  |
| NTRU+1152 | 248       | 266  | 222     | 233  |

Table 6: Concrete Security Level relative to LWE and NTRU problems

<sup>5</sup><https://github.com/malb/lattice-estimator/tree/352ddaf4a288a0543f5d9eb588d2f89c7acec463>

## 8 Performance Analysis

All benchmarks were obtained on a single core of an Intel Core i7-8700K (Coffee Lake) processor running at 3.7 GHz, with the benchmarking machine equipped with 64 GB of RAM. Implementations were compiled using gcc version 11.4.0. Table 7 lists the execution time of our optimized C and AVX2 implementations of NTRU+, as well as those of NTRU<sup>6</sup>, and KYBER<sup>7</sup>, together with the estimated security level and the sizes of the secret key, public key, and ciphertext. Execution time represent the average cycle counts over 100,000 runs of each algorithm. The source code for NTRU+ is available at <https://github.com/ntruplus/ntruplus>.

When comparing NTRU and NTRU+, Table 7 shows that the two schemes achieve similar bandwidth (public key plus ciphertext) at comparable security levels. For example, NTRU+864 at the 179-bit security level requires 2,592 bytes of bandwidth, while ntru4096821 at the 171-bit security level requires 2,460 bytes. In terms of secret-key storage, however, NTRU+ requires almost twice the cost of NTRU, because it stores  $(f, h^{-1}, F(pk))$  as the secret key, whereas NTRU stores only  $f$ . On the other hand, with respect to execution time, NTRU+ outperforms NTRU, largely due to the use of NTT-friendly rings.

When comparing KYBER and NTRU+, the bandwidth of NTRU+ is slightly larger than that of KYBER at comparable security levels. This is because KYBER uses a rounding technique to reduce the ciphertext size. In terms of efficiency, Table 7 shows that, at similar security levels, the key generation, encapsulation, and decapsulation of NTRU+ are all faster than those of KYBER in both the optimized C and AVX2 implementations.

---

<sup>6</sup>We use the code submitted to the NIST PQC Standardization Round 3.

<sup>7</sup><https://github.com/pq-crystals/kyber/tree/4768bd37c02f9c40a46cb49d4d1f4d5e612bb882>

Table 7: Comparison between the finalist NTRU, KYBER and NTRU+

| Scheme          | security level |         | $n$  | $q$  | $pk$ | $ct$ | $sk$ | $\log_2 \delta'$ | optimized C |       |       | AVX2 |       |       |
|-----------------|----------------|---------|------|------|------|------|------|------------------|-------------|-------|-------|------|-------|-------|
|                 | classical      | quantum |      |      |      |      |      |                  | Gen         | Encap | Decap | Gen  | Encap | Decap |
| NTRU+768        | 156            | 139     | 768  | 3457 | 1152 | 2336 | -379 | 99               | 75          | 78    | 27    | 31   | 19    |       |
| NTRU+864        | 179            | 160     | 864  | 3457 | 1296 | 2624 | -340 | 111              | 87          | 90    | 29    | 36   | 24    |       |
| NTRU+1152       | 248            | 222     | 1152 | 3457 | 1728 | 3488 | -260 | 176              | 118         | 126   | 44    | 45   | 30    |       |
| KYBER512        | 115            | 103     | 512  | 3329 | 800  | 768  | 1632 | -139             | 135         | 159   | 203   | 28   | 29    | 30    |
| KYBER768        | 174            | 155     | 768  | 3329 | 1184 | 1088 | 2400 | -164             | 232         | 258   | 318   | 45   | 44    | 46    |
| KYBER1024       | 241            | 215     | 1024 | 3329 | 1568 | 1568 | 3168 | -174             | 334         | 388   | 463   | 62   | 62    | 66    |
| ntruhaps2048509 | 104            | 93      | 509  | 2048 | 699  | 699  | 935  | $-\infty$        | 9296        | 624   | 1657  | 177  | 45    | 40    |
| ntruhrss701     | 132            | 117     | 701  | 8192 | 1138 | 1138 | 1450 | $-\infty$        | 17414       | 1060  | 3119  | 270  | 40    | 61    |
| ntruhaps2048677 | 142            | 127     | 677  | 2048 | 930  | 930  | 1234 | $-\infty$        | 16148       | 1071  | 2900  | 286  | 62    | 58    |
| ntruhaps4096821 | 171            | 152     | 821  | 4096 | 1230 | 1230 | 1590 | $-\infty$        | 21879       | 2003  | 4261  | 400  | 72    | 73    |

*n*: polynomial degree of the ring. *q*: modulus.  $(pk, ct, sk)$ : bytes.  $\delta'$ : worst-case (or perfect) correctness error.  
 (Gen, Encap, Decap): K cycles of optimized C or AVX2 implementations.

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## A Factoring the Trinomial

For a better understanding of applying the NTT, we describe the recursive factorization of the polynomial  $x^{1152} - x^{576} + 1$  in  $\mathbb{Z}_q[x]$ . Considering the initial Radix-2 NTT layer, the polynomial is factored as

$$x^{1152} - x^{576} + 1 = (x^{576} - \zeta^{144})(x^{576} - \zeta^{720}),$$

where  $\zeta$  is a primitive  $\ell = 864$ -th root of unity, and  $\zeta^{\ell/6} = \zeta^{144}$  is a primitive sixth root of unity modulo  $q$ .

Considering the Radix-3 NTT layer, these factors are further decomposed into lower-degree polynomials. For instance, the factorization of  $x^{576} - \zeta^{144}$  yields

$$\begin{aligned} x^{576} - \zeta^{144} &= (x^{192} - \zeta^{48})(x^{192} - \zeta^{48}\omega)(x^{192} - \zeta^{48}\omega^2) \\ &= (x^{192} - \zeta^{48})(x^{192} - \zeta^{336})(x^{192} - \zeta^{624}), \end{aligned}$$

where  $\omega = \zeta^{288} = \zeta^{\ell/3}$  is a primitive third root of unity modulo  $q$ .

Considering the subsequent Radix-2 NTT layers, these factors are further bisected. For example,

$$x^{192} - \zeta^{48} = (x^{96} - \zeta^{24})(x^{96} + \zeta^{24}) = (x^{96} - \zeta^{24})(x^{96} - \zeta^{456}),$$

where  $\zeta^{432}$  is a primitive second root ( $\zeta^{\ell/2} \equiv -1 \pmod{q}$ ). This recursive factorization continues until the polynomials reach the terminal degree  $d = 4$ , defining the final component rings of the NTT domain.

## B Radix-3 NTT layer

For a clearer understanding, we describe the Radix-3 NTT layer used in our implementation. The Radix-3 NTT layer establishes a ring isomorphism between  $\mathbb{Z}_q[x]/\langle x^n - \alpha^3 \rangle$  and the product ring  $\mathbb{Z}_q[x]/\langle x^{n/3} - \alpha \rangle \times \mathbb{Z}_q[x]/\langle x^{n/3} - \beta \rangle \times \mathbb{Z}_q[x]/\langle x^{n/3} - \gamma \rangle$ , where  $\beta = \alpha\omega$  and  $\gamma = \alpha\omega^2$  (with  $\omega$  representing a primitive third root of unity modulo  $q$ ). To transform a polynomial  $a(x) = a_0(x) + a_1(x)x^{n/3} + a_2(x)x^{2n/3} \in \mathbb{Z}_q[x]/\langle x^n - \alpha^3 \rangle$  (where  $a_0(x)$ ,  $a_1(x)$ , and  $a_2(x)$  are polynomials with maximum degree  $n/3 - 1$ ) into the form  $(\hat{a}_0(x), \hat{a}_1(x), \hat{a}_2(x)) \in \mathbb{Z}_q[x]/\langle x^{n/3} - \alpha \rangle \times \mathbb{Z}_q[x]/\langle x^{n/3} - \beta \rangle \times \mathbb{Z}_q[x]/\langle x^{n/3} - \gamma \rangle$ , the following equations must be computed.

$$\begin{aligned} \hat{a}_0(x) &= a_0(x) + a_1(x)\alpha + a_2(x)\alpha^2, \\ \hat{a}_1(x) &= a_0(x) + a_1(x)\beta + a_2(x)\beta^2, \\ \hat{a}_2(x) &= a_0(x) + a_1(x)\gamma + a_2(x)\gamma^2. \end{aligned}$$

Naively, these equations might appear to require  $2n$  multiplications and  $2n$  additions, relying on six pre-defined values:  $\alpha$ ,  $\alpha^2$ ,  $\beta$ ,  $\beta^2$ ,  $\gamma$ , and  $\gamma^2$ . Nevertheless, following the techniques in [18], this cost can be reduced to  $n$  multiplications,  $n$  additions, and  $4n/3$  subtractions, while using only three predefined values  $\alpha$ ,  $\alpha^2$ , and  $\omega$  as shown in Algorithm 14.

$$\begin{aligned} \hat{a}_0(x) &= a_0(x) + a_1(x)\alpha + a_2(x)\alpha^2, \\ \hat{a}_1(x) &= a_0(x) - a_2(x)\alpha^2 + \omega(a_1(x)\alpha - a_2(x)\alpha^2), \\ \hat{a}_2(x) &= a_0(x) - a_1(x)\alpha - \omega(a_1(x)\alpha - a_2(x)\alpha^2). \end{aligned}$$

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**Algorithm 14:** Radix-3 NTT layer

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**Require:**  $a(x) = a_0(x) + a_1(x)x^{n/3} + a_2(x)x^{2n/3} \in \mathbb{Z}_q[x]/\langle x^n - \zeta^3 \rangle$   
**Ensure:**  $(\hat{a}_0(x), \hat{a}_1(x), \hat{a}_2(x)) \in \mathbb{Z}_q[x]/\langle x^{n/3} - \alpha \rangle \times \mathbb{Z}_q[x]/\langle x^{n/3} - \beta \rangle \times \mathbb{Z}_q[x]/\langle x^{n/3} - \gamma \rangle$

- 1:  $t_1(x) = a_1(x)\alpha$
- 2:  $t_2(x) = a_2(x)\alpha^2$
- 3:  $t_3(x) = (t_1(x) - t_2(x))\omega$
- 4:  $\hat{a}_2(x) = a_0(x) - t_1(x) - t_3(x)$
- 5:  $\hat{a}_1(x) = a_0(x) - t_2(x) + t_3(x)$
- 6:  $\hat{a}_0(x) = a_0(x) + t_1(x) + t_2(x)$
- 7: **return**  $(\hat{a}_0(x), \hat{a}_1(x), \hat{a}_2(x))$

---

Considering the aforementioned Radix-3 NTT layer, we need to compute the following equations to recover  $a(x) \in \mathbb{Z}_q[x]/\langle x^n - \zeta^3 \rangle$  from  $(\hat{a}_0(x), \hat{a}_1(x), \hat{a}_2(x)) \in \mathbb{Z}_q[x]/\langle x^{n/3} - \alpha \rangle \times \mathbb{Z}_q[x]/\langle x^{n/3} - \beta \rangle \times \mathbb{Z}_q[x]/\langle x^{n/3} - \gamma \rangle$ .

$$\begin{aligned} 3a_0(x) &= \hat{a}_0(x) + \hat{a}_1(x) + \hat{a}_2(x), \\ 3a_1(x) &= \hat{a}_0(x)\alpha^{-1} + \hat{a}_1(x)\beta^{-1} + \hat{a}_2(x)\gamma^{-1}, \\ 3a_2(x) &= \hat{a}_0(x)\alpha^{-2} + \hat{a}_1(x)\beta^{-2} + \hat{a}_2(x)\gamma^{-2}. \end{aligned}$$

Naively, these equations might appear to require  $2n$  multiplications and  $2n$  additions, relying on six predefined values:  $\alpha^{-1}, \alpha^{-2}, \beta^{-1}, \beta^{-2}, \gamma^{-1}$ , and  $\gamma^{-2}$ . Nevertheless, by following the techniques in [18], we can significantly reduce this computational load to  $n$  multiplications,  $n$  additions, and  $4n/3$  subtractions, while using only three predefined values:  $\alpha^{-1}, \alpha^{-2}$ , and  $\omega$ , as described in Algorithm 15.

$$\begin{aligned} 3a_0(x) &= \hat{a}_0(x) + \hat{a}_1(x) + \hat{a}_2(x), \\ 3a_1(x) &= \alpha^{-1}(\hat{a}_0(x) - \hat{a}_1(x) - \omega(\hat{a}_1(x) - \hat{a}_2(x))), \\ 3a_2(x) &= \alpha^{-2}(\hat{a}_0(x) - \hat{a}_2(x) + \omega(\hat{a}_1(x) - \hat{a}_2(x))). \end{aligned}$$

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**Algorithm 15:** Radix-3 Inverse NTT layer

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**Require:**  $(\hat{a}_0(x), \hat{a}_1(x), \hat{a}_2(x)) \in \mathbb{Z}_q[x]/\langle x^{n/3} - \alpha \rangle \times \mathbb{Z}_q[x]/\langle x^{n/3} - \beta \rangle \times \mathbb{Z}_q[x]/\langle x^{n/3} - \gamma \rangle$   
**Ensure:**  $3a(x) = 3a_0(x) + 3a_1(x)x^{n/3} + 3a_2(x)x^{2n/3} \in \mathbb{Z}_q[x]/\langle x^n - \alpha^3 \rangle$

- 1:  $t_1(x) = \omega(\hat{a}_1(x) - \hat{a}_2(x))$
- 2:  $t_2(x) = \hat{a}_0(x) - \hat{a}_1(x) - t_1(x)$
- 3:  $t_3(x) = \hat{a}_0(x) - \hat{a}_2(x) + t_1(x)$
- 4:  $3a_0(x) = \hat{a}_0(x) + \hat{a}_1(x) + \hat{a}_2(x)$
- 5:  $3a_1(x) = t_2(x)\alpha^{-1}$
- 6:  $3a_2(x) = t_3(x)\alpha^{-2}$
- 7: **return**  $3a(x) = 3a_0(x) + 3a_1(x)x^{n/3} + 3a_2(x)x^{2n/3}$

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Note that we can reuse the predefined table used for the NTT in the computation of the inverse NTT.

$$\begin{aligned} 3a_0(x) &= \hat{a}_0(x) + \hat{a}_1(x) + \hat{a}_2(x), \\ 3a_1(x) &= (\omega\alpha^{-1})(\hat{a}_2(x) - \hat{a}_0(x) + (\hat{a}_1(x) - \hat{a}_0(x))\omega), \\ 3a_2(x) &= (\omega^2\alpha^{-2})(\hat{a}_2(x) - \hat{a}_1(x) - (\hat{a}_1(x) - \hat{a}_0(x))\omega). \end{aligned}$$